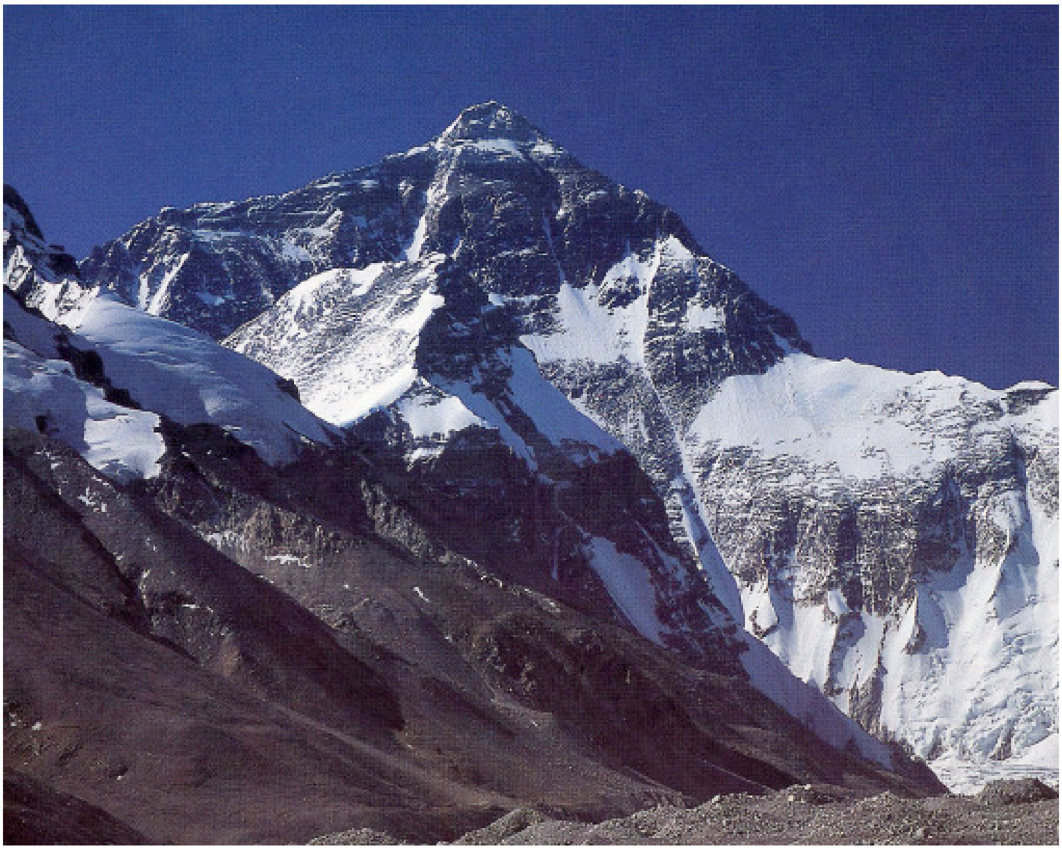
Science of Nature I

Mehmet Ali Alpar
SabancıUniversity, İstanbul

FALL 2011



Science of Nature I

Mehmet Ali Alpar
SabancıUniversity, İstanbul

FALL 2011

Edited, and with contributions by Defne Üçer-Şaylan and Yuki Kaneko

Any feedback, criticism, corrections and suggestions on the text, problems and questions will be highly appreciated for further improvement of the book. Please send an email to ducer@sabanciuniv.edu for comments.

Contents

1 A Brief Philosophical Introduction **2**

2 More on Science **5**

3 The Relation Between the Sciences **9**

4 Some Basic Tools: Scientific Notation, Errors, Dimensional Analysis, Fundamental Constants and Vectors **15**

4.1 Mathematics as a Language 15
4.2 Scientific Notation and Errors 16
4.3 Dimensional Analysis 18
4.4 Fundamental Constants 19
4.5 Vectors 20
4.5.1 Magnitude 21
4.5.2 Addition & Multiplication 21
4.5.3 Unit Vectors 22
4.5.4 Inner product 23
4.5.5 Vector product 26
4.5.6 Why are vectors useful? 26

5 How do we describe motion? **32**

5.1 Average velocity 33
5.2 How does one calculate dx/dt? 36

6 How to find out where the particle is? **42**

7 Circular motion **49**

7.1 Uniform circular motion 49
7.2 Position, velocity and acceleration in uniform circular motion 51

8 Newton's Laws of Motion **56**

8.1 Forces Acting on a Body 58
8.2 What is Weight? 60
8.3 Normal Forces 60
8.4 Friction 64

9 Momentum **70**

9.1 Collisions 72

9.2 * Center of Mass 73

10 Energy **77**
10.1 Non-Conservative Forces 80
10.2 Potential Energy 81
10.3 Elastic Collisions 86

11 * Damping and the Exponential Function **92**
11.1 Damping Forces Proportional to Velocity 92
11.2 The Exponential Function 93

12 The Harmonic Oscillator **98**

13 Angular Momentum **106**
13.1 Symmetries and Conservation Laws - Noether's Theorem 108

14 Gravitation and the Kepler Problem **110**

Exam Problems - Mechanics **114**

15 Pressure − Volume − Temperature: The Ideal Gas **130**

Exam Problems - Thermodynamics **136**

16 Coulomb's Law **138**
16.1 Fundamental Particles, the Quantum of Charge, and Our Units of Charge . . 138
16.2 The Electrostatic (or Coulomb) Force 139
16.3 The Electric Field and Voltage 143

17 Gauss' Law **150**

18 Parallel Plate Capacitors **158**

Exam Problems - Gauss' Law **163**

19 Magnetic Fields, Electric Currents and Ampere's Law **167**
19.1 Magnetism 167
19.2 Currents Make Magnetic Fields: Ampere's Law 167
19.3 * Contribution of Pieces of Current to the Magnetic Field: the Biot-Savart Law 170
19.4 Effect of a Magnetic Field on Electric Currents: the Lorentz Force 172

20 Cutting Electric and Magnetic Dipoles **179**

21 Changing Electric Fields Makes Circulating Magnetic Fields: the Displacement Current **183**

22 Faraday's Law of Induction **187**

23 Waves from Maxwell's Equations **192**
23.1 Application of Ampere's Law 193

23.2 Application of Faraday's Law 194
23.3 Obtaining the Wave Equation 196
23.4 Electromagnetic waves propagate with the speed of light: 196
23.5 Energy Radiated in Electromagnetic Waves 199

24 * Circuits **202**
24.1 Ohm's Law 203
24.2 Work, Energy, Power 205
24.3 Energy Stored in the Electric Field 205
24.4 Energy Stored in the Magnetic Field 208
24.5 Energy in Electromagnetic Waves 209

Exam Problems - Electricity and Magnetism **213**

25 Bohr's Model of the Hydrogen Atom **223**
25.1 Classical Physics: Why Are Atoms Like This? How Can Atoms Exist? 223
25.2 Bohr's Model: What Chooses the Size of the Atom? 225

26 Wave-Particle Duality **230**
26.1 Particle Properties 231
26.2 Wave Properties 231
26.3 Dictionary of Wave and Particle Properties 233
26.4 What is the Meaning of the Particle Wave? 234
26.5 The Schrödinger Equation 235
26.6 Fundamental Constants, Fundamental Forces and Wave-Particle Duality . . . 240

27 Wave Packets and the Uncertainty Relation **243**
27.1 Wave Packets 243
27.2 The Uncertainty Relation 244
27.3 Zero-Point Motion: Nothing Can Sit Still 246

28 The Pauli Exclusion Principle **252**

29 Wave Shapes for Atomic Electrons **260**

Exam Problems - Quantum Mechanics **263**

Answers to Selected Problems **269**

Glossary **277**

Fundamental Physical Constants

Quantity	Symbol	Value*	Units
Speed of light in vacuum	c	299 792 458	m s^{-1}
Magnetic constant	μ_0	$4\pi \times 10^{-7}$	N A^{-2}
(the permeability of free space)		$= 12.566\ 370\ 614... \times 10^{-7}$	N A^{-2}
Electric constant $1/\mu_0 c^2$	ϵ_0	$8.854\ 187\ 817... \times 10^{-12}$	N^{-1} m^{-2} C^2
Coulomb's constant $1/4\pi\epsilon_0$	K, k_e	$8.987\ 551\ 787... \times 10^9$	N m^2 C^{-2}
Gravitational constant	G	$6.674\ 28(67) \times 10^{-11}$	m^3 kg^{-1} s^{-2}
Planck constant	h	$6.626\ 068\ 96(33) \times 10^{-34}$	J s
$h/2\pi$	$\hbar$	$1.054\ 571\ 628(53) \times 10^{-34}$	J s
Elementary charge	e	$1.602\ 176\ 487(40) \times 10^{-19}$	C
Electron mass	m_e	$9.109\ 382\ 15(45) \times 10^{-31}$	kg
Proton mass	m_p	$1.672\ 621\ 637(83) \times 10^{-27}$	kg
Proton-electron mass ratio	m_p/m_e	1836.152 672 47(80)	
Fine-structure constant $e^2/4\pi\epsilon_0\hbar c$	α	$7.297\ 352\ 5376(50)\ \times 10^{-3}$	
Inverse fine-structure constant	α^{-1}	137.035 999 679(94)	
Avogadro constant	N_A	$6.022\ 141\ 79(30) \times 10^{23}$	mol^{-1}
Boltzmann constant	k, k_B	$1.380\ 6504(24) \times 10^{-23}$	J K^{-1}
Molar gas constant $k_B N_A$	R	8.314 472(15)	J mol^{-1} K^{-1}
Electron volt $\equiv e$ J/C	eV	$1.602\ 176\ 487(40) \times 10^{-19}$	J
(unified) atomic mass unit			
1 u $= m_u = \frac{1}{12} m(^{12}C)$	u	$1.660\ 538\ 782(83) \times 10^{-27}$	kg

*Values in parentheses are uncertainties corresponding to the last two digits. For example, $6.674\ 28(67) \times 10^{-11}$ means $(6.674\ 28 \pm 0.000\ 67) \times 10^{-11}$.

List of Symbols Used in This Book

Symbol	Definition
x, y, z, r	Position
v	Velocity
a	Acceleration
t	Time
m	Mass
Q, q	Charge
T	Temperature, Tension, Period
F	Force
g	Gravitational acceleration on the Earth (= 9.81 m/s)
θ (*theta*)	Angle
ω (*omega*)	Angular velocity
P	Period, Pressure, Power
p	Momentum
μ (*mu*)	Coefficient of friction
W	Work, Weight
U	Potential energy
K, KE, E_K	Kinetic energy
L	Angular momentum, Inductance
τ (*tau*)	Torque
k	Spring constant, Wave number
V	Volume, Voltage
A	Area
E	Energy, Electric field

Symbol	Definition
j	Current density
λ (*lambda*)	Wavelength
f	Frequency
ϕ (*phi*)	Phase constant
I	Current
σ (*sigma*)	Charge density
C	Capacitance
B	Magnetic field
Φ (*phi*)	Flux, Wave function
R	Resistance
Ψ, ψ (*psi*)	Wave function

Chapter 1

A Brief Philosophical Introduction

Science is a particular organized way of learning about the world – the Universe, all of Nature, taken to include also the little corner of the Universe that is occupied by human beings, their minds, cultures and societies. Science is a method of acquiring knowledge, but scientists themselves rarely go through the rules of the scientific method consciously and explicitly. Nevertheless the method is fundamental to science. It is taken as common sense by scientists. Indeed the scientific method is a specialized extension of common sense in daily life. The development of science itself has had a tremendous impact on civilization, not only through its practical results as technology, but philosophically, as an embodiment of a common sense approach to the world.

So what are the basic philosophical *beliefs* of scientists? First there is an external / material world. By "world" I mean the Universe and everything in it, not our Earth alone. "External/ material" means our knowledge of the world is based on something outside our minds from which common knowledge can be derived by different people, different minds.

Knowledge of the world is possible by *observation* and *experiment* plus logical deductions (mathematics). The experimental basis of science was very explicitly laid out and used by the pioneers of modern science, like Galileo Galilei and others about whose work we will learn later in the course. The first great example of the formulation of laws of nature by using mathematics was the work of Newton – who also developed a whole area of new mathematics, the calculus, to be able to express and use these laws. This course will start with mechanics, the knowledge of how things move, as Galileo and Newton developed it, along with learning and using the calculus.

Objectivity is a word that is often associated with science. Scientists as persons are of course no different from other human beings, they all have their emotions and ambitions. Scientists' motivations for doing science like other aspects of their lives show a lot of variation. The term objectivity refers to something fundamental for the possibility of science, and indeed for common sense and practical knowledge to be possible at all: different observers find the same results from the same experiment under the same conditions.

The scientific method follows a flowchart like this: Prediction → experiment → verified?
→ one more check; not verified?
→ hypothesis is wrong.
OR
→ Theory is limited, not correct for the new conditions of this experiment: so it is wrong as a complete theory. A better, more general theory must be found.

As the philosopher David Hume realized, finding the same result in the same experiment

N times does not prove that the $(N+1)th$ time the same result will be found again.

It is the fundamental philosophical *belief* of the scientist that on the $(N+1)th$ time the same experiment will give the same result. This has been verified so far by all the successful experience of science itself.

The practice of science and of course our daily experience support this belief. We try the world out, and deduce logically from our experience. We learn what works/holds is "true" by experiment and logical deduction. Science is systematic common sense.

So scientific results are not verified ever in an absolute sense. The next time of repeating the experiment, under exactly the same conditions, might just give a different result: we cannot know this before doing the experiment. But so far in the entire history of science and of human experience in general a result that has been repeatedly observed under the same conditions a large number of times has come out the same way every next time the observation and experiment was repeated. Science is subject to verification in this strong but still in principle limited way. Nevertheless it is subject to verification, and falsification in a way that other ways of attaining knowledge, or beliefs of different sources are not.

The 20th century philosopher Karl Popper emphasized that the potential for the converse of verification, falsification, is an even stronger and rigorous characteristic that distinguishes scientific knowledge. Scientific knowledge is *falsifiable.* Just one instance of careful experimental falsification is enough to discredit a scientific proposition false. It has never happened that in exactly the same circumstances the same testable claim is sometimes true and sometimes false. But it has happened that the predictions of a scientific theory that was tested and verified in very many different circumstances turns out not to hold in a new and different circumstance that was not tested before. When this happens the old theory is falsified as a *general* theory that explains everything. It becomes a limited theory that is valid under many but not all circumstances. This points the way to a search for a new, more general theory that explains the new experiments, but still agrees with the old theory, to a good approximation, in the situations of the old experiments. Newtonian mechanics is a good example of a theory that is very successful but still does not cover all situations. Experiments involving motion at very high speeds, close to the speed of light, and experiments showing that the speed of light has the same value as measured by all observers, no matter how fast these observers are moving with respect to each other have falsified Newton's Mechanics as a general theory. The (so far not falsified) successful new theory was Einstein's Special Theory of Relativity, and his General Theory of Relativity, which treats gravitation and acceleration. Another modern theory which replaces classical physics in the realm of the atom and the structure of matter is Quantum Mechanics. This is a very strange theory in terms of our familiar daily experience. The world is fundamentally probabilistic according to Quantum Mechanics, the questions we can test experimentally are about probabilities. But this strange property of the world does not change the scientific method, the questions of quantum mechanics are tested by experiments in the same way that those of other sciences are, and so far quantum mechanics has not been falsified. Another very successful theory of modern science is Darwin's theory of evolution. This has also not been falsified yet in any of its predictions, and has led to practical results that we are using as technological products, through the understanding of genetics and in agreement with the results of molecular biology. The theory of evolution has still many questions that have not been fully understood. The interface of quantum mechanics and general relativity is another frontier where two very successful theories are still incomplete, facing very difficult problems. But this is the nature of science: it is never complete, the search never ends. A good scientific theory has just not been falsified in all the realms of experiment where it has been tested so far. A scientific theory can never be really final and complete. But it has been tested and not falsified yet. It may or may not be falsified in the future. It is open to more testing and potential falsification. This is perhaps as true as 'true' can be, in a sense that can be tested, learned and used by all.

It is surprising but true that what we discover from experiment and observation can be described in terms of *laws of nature*; laws of nature can be expressed mathematically and the basic laws are simple and beautiful.

This is surprising but true! In the words of Galileo, according to legend:
"Eppur si muove!" – "and yet it moves!"

It is not known whether Galileo actually said these words. He would be referring to the Earth, but these words can well be applied to the observation, and marvel, that Nature moves in ways that we can understand.

Science says something only about the observable world. It does not prove or disprove any religion or philosophy as long as the religion or philosophy is not interpreted to make specific, concrete statements about the world. In such cases science has shown that such literal religious beliefs have no foundation in the observed world.

Chapter 2

More on Science

Scientific statements are statements about the world that can be *falsified* by experiment. Science accepts no unchangeable truths. It also does not accept unchecked a priori judgments, beliefs or guesses, no matter how reasonable they may seem. Reasoned guesses are hypotheses that are to be checked by experiment. A scientific *"truth"* is something that has been checked and verified a very large number of times by a large number of independent observers. *This is as 'true' as true can be.*

Scientific Knowledge draws its strength from the fact that it is testable, not to be taken for granted or just accepted. Anyone repeating the experiment or observation under the same conditions gets the same result.

Scientific statements are *conditional.* Scientific knowledge is ultimately derived from and checked by observations and experiments. Like the results of experiments they are derived from, all scientific statements depend on the conditions, coordinates, constraints of the system being observed or experimented on. *"Conditional" does not mean "arbitrary".*

Experimental results (measurements) are stated with their possible errors (accuracy) and sources of error.

Example: Objects *near the surface of the Earth* fall down – they are accelerated towards the Earth – with an acceleration that has the approximate value $g = -9.81$ m/s^2 *on the surface.* The conditions here are stated in *italics.* One can be more specific for a more accurate description of the experimental conditions: What does "*near*" mean? What does "*on the surface*" mean? Is g the same at different latitudes and longitudes?

The Scientific Method rests on experiment and observation as the only ways of getting knowledge about the natural world. This coincides with common sense in how we conduct our lives in this world. I know, by my own and other people's experience that if I drink dirty water I can get ill. So I do not drink dirty water. Common sense is built on experience, and not only on one's personal experience, but by experience shared by other people. Experience can be shared by other people because in the natural world the same thing happens every time, and to anyone, under the same conditions. The Scientific Method is a systematic extension of common sense. First it is careful to note the conditions and control the experiments and monitor the observations. Secondly, Science observes the natural world under conditions far beyond our common daily experiences. Some of the important pieces of scientific knowledge seems or seemed at first to be actually against common sense. These may be against *common daily experience*, but not against common sense as employed in Science.

Famous examples of scientific propositions that seemed or seem to be against 'common sense' are the propositions that:

- Iron and cotton fall down in exactly the same way in vacuum.
- The Earth is not flat.
- The Earth moves through space in orbit around the Sun.
- An electron, an atom and in fact any object is both a wave and a particle (Quantum Mechanics).
- If one of two twins takes a space trip, accelerates to speeds close to the speed of light, turns around and returns to the Earth, he will be younger than his twin left on Earth (Relativity).
- Human beings and monkeys descend from the same ancestor species that lived millions of years ago (Evolution).

Scientific knowledge that seems incredible and fascinating is verified with experiments and observations in realms of space and time and under conditions that are inaccessible to our daily experience. Things that happen very very rarely can find a chance to happen given long enough times. Science finds out if, how and why its seemingly incredible hypotheses are *true*, in the scientific sense of *falsifiable, tested many many times and not falsified so far*. Some of the many fascinating hypotheses that the human mind has come up with do turn out to be true. Nature can indeed be stranger than fiction, and the Science of Nature, the process of finding out, can be as fascinating as Nature itself.

A *theory* in science is based on a large number of experiments covering different conditions. A theory may consist of some general *laws* and specific rules, along with the *logical connections* between them. A theory makes *predictions*, which are *hypotheses* to be tested by new *experiments* (But all hypotheses do not come from an existing general theory).

In daily language 'theory' can mean 'guess', 'speculation'. Theory has a different, *well defined* meaning in science:

> A theory is an organized body of knowledge which can make predictions, and whose predictions are already well tested by experiment.

Newton's mechanics, Maxwell's electromagnetic theory, the laws of thermodynamics, Einstein's theory of relativity, Darwin's theory of evolution, the atomic theory of matter and quantum mechanics are all scientific theories.

Sometimes a general, fundamental, well tested theory conflicts with *new* experiments. This happens rarely, usually as a result of new technology making new experiments possible. When it happens the conditions under which the old theory is valid become restricted. The old theory is no longer a complete general theory, and is no longer correct under all circumstances. As a general theory it is found out to be wrong. But it is still correct and useful for all the situations where it was tested and verified before. The old theory must now be replaced with a new, more general theory. The new theory must include the old theory under restricted conditions but must generalize the old theory to new situations. The new theory now makes new predictions and gains in success to the extent that these predictions are verified by experiment.

Example: Newton's mechanics does not hold for motions which involve velocities near the speed of light! The new, more general theory is Einstein's theory of relativity. This theory makes predictions that are verified in experiments involving motion at speeds close to the speed of light. At small velocities $v \ll c$ Einstein's predictions agree with Newton's mechanics to high accuracy.
According to Newton's mechanics if an object is moving with respect to observer 1 with velocity $\mathbf{v}$ and observer 1 is moving with respect to observer 2 with velocity $\mathbf{V}$ then the object is moving with velocity

$$\mathbf{v}' = \mathbf{v} + \mathbf{V}$$

with respect to the observer 2. This seems obvious, as it is part of our daily experience and borne by the results of many experiments.
Einstein's theory predicts instead that

$$\mathbf{v}' = \frac{\mathbf{v} + \ \mathbf{V}}{1 + \mathbf{v} \cdot \mathbf{V}/c^2}$$

where c is the speed of light. At low speeds that we are used to the old result is accurate enough. It differs from the exact result of Einstein by an error of order vV/c^2 , which is very small for the situations we are used to or the experiments of classical mechanics. But experiments at high velocities agree with Einstein's prediction. In particular, the velocity of light c is the same with respect to all observers no matter how fast they are moving. No object can move faster than light.
Einstein took his clue from the simple and beautiful theory of electricity and magnetism formulated by Maxwell, which seemed to be in conflict with Newtonian mechanics. It turned out that Maxwell's equations form a correct general theory that is consistent with Einstein's relativistic mechanics. When Einstein developed his theory he was not yet aware of the Michelson-Morley experiment which failed to detect any dependence of the velocity of light on the velocity of the Earth.

CHAPTER 2 - PROBLEMS:

1. Calculate the gravitational acceleration
   (a) On the surface of the Earth,
   (b) On top of the World's tallest building (the Taipei financial center in Taiwan – 509 m),

(c) On top of Mount Everest (peak $\sim 8700\ m$).

$$g = \frac{GM_\oplus}{(R_\oplus + H)^2}$$

Here g is the gravitational acceleration, G is the universal gravitational constant, $M_\oplus \cong 6 \times 10^{24}\ kg$ is the mass of the Earth, $R_\oplus \cong 6400\ km$ is the radius of the Earth and H is the height from the surface of the Earth. The Earth is assumed to be perfectly spherical. (Is this correct?)

2. The gravitational acceleration g near the surface of the Earth,

$$g = \frac{GM_\oplus}{R_\oplus^2}(1 + \frac{H}{R_\oplus})^{-2}$$

can be expanded approximately using the *Binomial Expansion Formula*

$(1+x)^k \cong 1 + kx + \frac{k(k-1)}{2}x^2 + ...$

which is valid for any exponent k, not necessarily an integer. For $x << 1$, the expansion gives a good approximation with only two or three terms.

Evaluate g on top of Mount Everest using the Binomial Expansion Formula, and compare with your result in Problem 1.

3. You are driving on a highway at a speed of 100 km/h. You see another car is passing by at 20 km/h with respect to you.

a) How fast does the other car travel with respect to a stationary observer? Can you find the answer by Newton's theory? How different is it from Einstein's theory?

b) The light from your headlights travels with the speed of light c with respect to your car. How fast does the light from your headlights travel with respect to a stationary observer according to Newton's Mechanics? What is the answer according to Einstein's theory?

Chapter 3

The Relation Between the Sciences

When we study a system in the natural world, say in biology, geology or chemistry, the frequently asked, basic questions are:

> *"What is it made of?"*
>
> *"How does it work?"*
>
> *"Why?"* in the sense of : "*What causes lead to these effects?*" Not in the sense of "*For what purpose?*" – this is not a scientific question. (Why not?)

These questions, asked again and again, lead to:

- What are the smallest building blocks of matter – atoms, nuclei, elementary particles?

  Are these made of other things? Is there an end to the list of smaller and smaller particles? If so what are the final basic building blocks? Is the present list of quarks and leptons the complete list? Do the fundamental particles expected by present theories exist? What are their masses, charges and other properties? Scientists are now looking for answers with the huge experiment machines at CERN.

- What are the Laws of Nature – which are the more basic laws – which are the fundamental forces from which all interactions between pieces of matter and energy can be derived?

  Are these the most basic forces? Can they be derived from something even deeper?

  Are some of the four basic forces of physics (gravity, electromagnetism, weak interactions, strong interactions) really aspects of the same thing, so that there really are fewer (unified) basic forces? Present theories indicate that the electromagnetic, weak and strong forces are aspects of one unified basic interaction. This is already verified by experiments to a great extent, but all details have not been tested yet, and there are still remaining important questions to be checked out.

  But with three fundamental forces unified, gravitation is left out. Scientists do not know how gravity relates to the other forces. We do not yet know how gravity relates to quantum mechanics. Gravity is a very weak but long range force, it acts between all pieces of mass and energy. Gravity governs the motion of largest bodies, the Earth, planets, stars, galaxies. Gravity also governs the history of the whole Universe from the Big Bang on.

Since the early twentieth century there has been a tremendous development in Cosmology, the science of the Universe. The Universe as a whole has become a subject of scientific enquiry. We have learned from observations that the Universe is expanding and must have come from a Big Bang. It now seems like the Universe has been expanding faster and faster as time goes on, and gravity will never be able to stop and pull back this expansion.

- A type of question that comes up frequently in scientific research is *"What do I need to understand first to understand the problem I have at hand?"*.

Such an approach leads to reducing problems in biology to chemistry and then to problems of microscopic physics. It is called *reductionism*. It works as a principle, and allows us to see the overall structure of science as a whole.

This was not obvious historically but learned, from experiment, in the course of the history of science. Many important questions of science have been explained with a reductionist approach. Some examples are:

- *Do organisms function on the basis of the same laws of mechanics that govern inorganic matter?* One important answer to this question came first in the 17th century when Harvey realized that blood circulates under pressure, with the heart acting as a pump.
- *Is living matter made of the same stuff as inorganic matter ?* This was answered by the 19th century development of organic chemistry. The synthesis of urea in the lab was a very important step in this direction.

But reductionism is not the whole story. At different levels of quantity, or of qualitative differences, systems have different kinds of properties. New concepts are needed, new questions are asked in terms of these new concepts. For example *temperature, pressure, and entropy* are all concepts that have no meaning for a single particle, but are most useful for systems with large numbers of particles, of the order of Avogadro's number (6×10^{23}).

Very important laws, the *Laws of Thermodynamics*, are expressed in terms of these new concepts. These laws are scientific – they make predictions that can be tested by experiment. The relation between these macroscopic laws and the microscopic laws of physics is an interesting area of physics.

Other new concepts at new levels are, for example, concepts like *acid, base, solution, pH* in chemistry. An important question special to biology that is still unanswered is: *How did life start?* Another biology question that has answers in the theory of evolution, is *How do life forms vary?* (Do they vary or are species fixed for ever? That question has already been answered: yes, species vary, they evolve, branch out to new species and become extinct.) *How does heredity work?* This is answered in terms of genes and DNA molecules. In principle the mechanisms of genetics can be reduced to chemistry and physics, but this is not always practical, or useful.

In psychology many questions about *behavior, intelligence, mood* can be addressed in terms of the *physiology and biochemistry of the nervous system and the brain.* Again the successful working assumption is reductionist but the concepts are rooted at structures and functions of the nerve cells, for example. And there is much, much more to be learned about consciousness compared to the little that is known already.

A basic question in astronomy is *whether matter is made up of the same constituents and whether the same laws of nature apply everywhere in the universe?* The answer is yes, so far. Thus astronomy, like the other sciences, has a reductionist tradition. At the same time, astronomical observations find specific structures and dynamics of matter, governed by gravity, at large scales. New concepts are dictated by the structure of the universe at different distance scales. A very important development in Science is the observational evidence that matter at the largest scales has its own properties and dynamics – *Cosmology has become an observational science.* The Universe as a whole has become a subject of scientific enquiry. We have learned from observations that the Universe is expanding and must have come from a Big Bang. It now seems like the Universe has been expanding faster and faster as time goes on, and gravity will never be able to stop and pull back this expansion. Properties of matter that we observe from the farthest reaches of the Universe can be explained by the same laws of Nature that we have learned in our laboratories here on Earth. Along with this reductionist element, the last few decades of astronomy and cosmology indicates that ordinary matter makes up only about 5% of the Universe, the rest being dark matter (20%) whose nature we do not really know – it could be made up from several kinds of matter that are not easy to observe from far because they do not emit radiation – black holes or very small bodies, brown dwarfs, that did not light up as star, or some types of elementary particles. We do not know which of these candidates contribute, and how much of the dark matter. More mysterious still is dark energy. If the observations suggesting the accelerated expansion of the Universe continue to be confirmed, about 80% of the Universe is supposed to be dark energy, something that provides a repulsive force to counteract gravity. We have no idea what it is yet. Dark matter and dark energy are good current examples of *emergent* concepts in science.

Problems in science have different levels of *complexity* reflecting the complexity of the systems they study. Complexity is much more than just a large number of components in a system. In fact the appropriate laws for a large system at the macroscopic level and at least some types of behavior may not be complex at all. Complexity also involves the ways of interaction, the connections, between the parts of a large system. With the advent of computers it has become possible to model and classify certain types of complexity. Science is now able to revise the relations between its different disciplines and the relations between different sizes and scales in Nature across the board. The traditional reductionist approach is complemented by the study of *emergence*, how novel structures and functions emerge in large systems: the emergence of properties of the large system that are not meaningful for the small parts of the system.

There are certain laws of science that are relevant and valid at all levels, in all kinds of microscopic and macroscopic systems, with reductionist or emergent concepts. The conservation of energy is perhaps the most important example. Other conservation laws like the conservation of momentum, angular momentum, charge and mass also hold very generally.

Questions:

- What are some examples of emergent properties in Nature, as studied in different branches of Science?
- What are the emergent properties of water and air?
- How many water molecules exist in 1 lt of water?
- How many air molecules exist in a room with size $3m \times 3m \times 3m$?

Example: *Fluid mechanics* When we consider a fluid, we do not need to take into account the motion of every single molecule and its interaction with other fluid molecules and surroundings in order to understand the physics.
Instead we can make use of some *macroscopic quantities* which take into account a system of many particles. These macroscopic properties come out not from the behavior of a single particle or a molecule, they come out from a collective behavior of many particles.
Reynolds' number of a flow is one of these properties.

$$R = \frac{\rho V L}{\nu}$$

where V is the velocity of the fluid flow, L is the characteristic scale of the flow, ρ is the density of the fluid and ν is the viscosity. Viscosity is the measure of 'thickness' of a fluid. For instance honey has higher viscosity than water and water has higher viscosity than air. Reynolds' number says a lot about a fluid flow. One can determine whether a flow is turbulent or not, by looking at the Reynolds' number.
Turbulence is due to creation of disturbances (little fluctuations) in the flow which have much smaller scales than the flow itself. Disturbances can be in the form of vortices. This is due to transfer of energy to smaller scale dynamics. Little vortices form and slow down the flow. Experimentally it is known that for Reynolds' number > 1000, some turbulence is expected.

Not all sciences and not all problems in science are open to designed experiments. Instead the scientific enquiry has to proceed by observation. Observation is the only way in astronomy and geology. Experiments may not be possible or may not be acceptable on ethical grounds in biological and medical sciences, in physical sciences that can effect the environment, and in most areas of social sciences.

Social sciences are in general much more complex than the natural sciences. The *subjectivity* that is inherent makes it difficult to reduce the complexity by identifying the concepts and laws that are appropriate to the problem. Model building and analogies can be dangerous and misleading. A subjective element may be unavoidable in social sciences. To take account of subjectivity while acquiring knowledge and to record the viewpoint of the observer in a way that makes the communication of knowledge possible and the results understandable by others poses problems. This is a distinctly different dimension of difficulty that distinguishes social sciences from the complexities in natural sciences. But it is not justified to generalize from the particular difficulties and complexity of the social sciences to the false conclusion that science in general is undecidable, uncertain, and that all means of acquiring knowledge and all descriptions of the world are equally valid. It is also not justified to impose on one area of science the concepts and analogies from another area; in particular to expect models that work in the natural sciences, to carry over to social sciences. Analogies may provide helpful hints and inspiration, but they have to lead to falsifiable hypotheses, observable predictions or at least discussable consequences in order to be useful in a different context. The common goal of science, of acquiring verifiable and falsifiable and thereby sharable knowledge of the world is an ideal of all science, with extra complexities particular to social sciences.

Think of the analogies made between the changes of society and motion of a physical system or a machine. A statement that history is “cyclic”, or that there is always “progress”, may be expressed poetically, and be functional politically, and can contain insight, but it is not scientific in that it does not hold falsifiable, testable predictions. A falsifiable version could be attempted by defining “cyclic” or “progress” in some observable way. But then the theory may not be as general or as interesting. By contrast, a theory that the universe will have an end or that it will not is in principle (and probably in practice in the next decades) testable by certain measurements that can check the predictions of the theory about the state of the universe as observed now.

Other creative activities like mathematics, arts, technology share with science the property that they deal with their content with their *logic, intelligence and methods.* They also explore, learn by experience and propagate an accumulated culture. But they do not do this by the specific, structured way of the scientific method.

Nor do scientists employ the scientific method consciously all the time, in a step-by-step, follow-the-instructions way. They act with their personal or group motivations, just like everyone else. The scientific method nevertheless organizes the accumulated Science of Nature because nature has its laws. In the words attributed to Galileo, which he probably never actually said, “Eppur si muove.”

To quote Feynman, to say that something is not a science means only that it is not a science. It does not mean, by any means, that it is not good. In Feynman‘s words, “After all *love* is not a science.”

CHAPTER 3 - PROBLEMS:

1. Estimate the Reynolds’ number for blood flow in a major artery like the aorta. The radius of the aorta is ~ 1 cm, the mean flow velocity is ~ 0.35 m/s, density of blood at 37°C (body temperature) is $\sim 1000\ kg/m^3$, and the viscosity of blood is about 4 times the viscosity of water, which is $2.8 \times 10^{-3}\ kgs^{-1}m^{-1}$.
   Can blood flow be turbulent? Why is this important?

2. Are the following statements scientific or non-scientific? Why?

   (a) Eating an apple a day prevents catching a cold.

   (b) The Earth orbits around the Sun.

(c) Dogs are cute.

(d) The Carbon and Oxygen nuclei in our bodies were made in stars.

(e) A water molecule is made of Hydrogen and Oxygen.

(f) A water molecule is made of Nitrogen and Sulphur.

(g) The chimpanzee genome differs from the human genome by 1.23%.

(h) Ahmet likes cats.

(i) People from the zodiac sign Aries like adventure.

(j) Far away galaxies have less iron than our Milky Way.

3. Make up some examples of scientific and non-scientific statements.

Chapter 4

Some Basic Tools: Scientific Notation, Errors, Dimensional Analysis, Fundamental Constants and Vectors

4.1 Mathematics as a Language

The laws of nature and formulas of science are usually expressed as mathematical equations. Mathematics provides the language for expressing scientific information in a short and exact expression. People who are not familiar with science are sometimes put off, and even scared by the mathematical expressions. It will help if you recognize that all mathematical expressions are really sentences in a special language.

The syntax of the mathematical language is standard and simple. An equation like $A = B$ just states that the value of the quantity expressed by the symbol A equals the value of B. If one knows what A and B stand for the sentence is translated into ordinary language, and one can then think about what it says and try to understand it. Sometimes the relation between A and B is something other than equality, for example A might be greater than B, which is stated as $A > B$. The symbol between A and B is the verb of the mathematical sentence. Just as $=$ reads "is equal to", $>$ reads "is greater than" and so on.

Usually the right side of the expression is not a single symbol like B, but a combination of symbols and operations, like $A = B+C$, typically some complicated set of operations that may depend on quantities $B, C, D, ...$ in which case we have an expression like $A = f(B, C, D, ...)$ which reads " A is equal to *a function of* the quantities B, C, D " . The function (or dependence) will involve operations like addition, multiplication, various functions like taking the square, the fifth power, or things like the *sine*, *cosine*, *exponential*, or *logarithm*.

Do not worry if you have not learned these things before or you have forgotten what they mean. When the time comes we will learn them in the context of a science subject. The important first thing now is to feel comfortable with mathematical sentences, and to read equations and translate them into "A is equal to something calculated in this particular way from B, C etc." The symbols like A, B, C are the nouns in the sentence – they are just short names for things. Once you know what the symbols stand for, you are done: you can read the mathematical expression like a sentence.

A famous example of a Law of Nature is Newton's Second Law, expressed with the formula $F = ma$, which reads "F equals m times a. In science usually particular symbols are used always for the same quantities, like x, y and z are used to denote the position of an object. Thus F is the usual symbol for force, m for mass, and a for acceleration. So Newton's

Second Law reads "the force exerted on an object is equal to the product of the mass of the object and the acceleration of the object." This is the translation of the mathematics into words. To understand the statement, of course one has to understand what force, mass and acceleration mean. This is what we will be learning as we go along: we will be learning what the concepts in science mean in terms of the experiments and measurements defining them.

When a new symbol is first defined in terms of other symbols we will use the notation $\equiv$ instead of $=$. Thus $A \equiv f(B, C, ..)$ reads A is defined as the short symbol for the expression $f(B, C, ...)$. In "List of Symbols" in the front matter of this book, we list some of the common symbols and relations, and what they stand for.

Mathematics also supplies a very powerful visual language: this is geometry, particularly the use of graphs in science. The rule here is simple: a graph expresses a relation like $A = f(B)$ visually. One labels the vertical axis as A and the horizontal axis as B, and writes the units for A and B on the axes. The relation $A = f(B)$ appears as a curve in the graph. To graph observational or experimental data, one marks the data points with measured numerical values of (B, A) on the graph, together with their *error bars* as we discuss below. Figure 4.1 gives a simple example of a graph, the distance $y = 1/2gt^2$ that an object released from rest falls down in time t: the formula is shown as a curve, and some data points obtained by experiment are also shown together with their error bars.

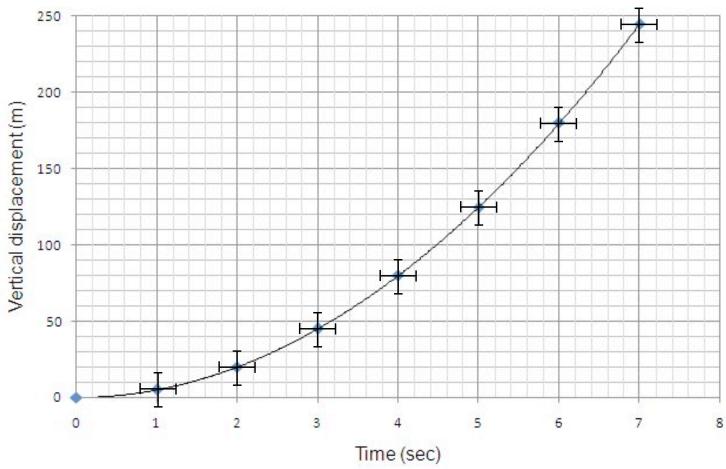


Figure 4.1:

While many parts of mathematics have been and still are started by developments in science, especially physics, mathematics is much more than just the wonderful language it provides for science. Many areas of pure mathematics develop independently of science. Sometimes an abstract area of mathematics may suddenly become useful as language to express the results for some area of science.

4.2 Scientific Notation and Errors

Nature has many different scales. Lengths (sizes) vary from the 10^{-15} meters or smaller scale of the fundamental particles like neutrons, protons, electrons and quarks, to the dimensions of the Universe, 10^{26} meters. Timescales vary from as short as, for example, the 10^{-27} second period of oscillation for very high frequency electromagnetic waves (which are very high energy *γ-ray photons*), up to the age of the Universe, $\sim 10^{10}$ yrs. A wide range of scales and values is encountered in Nature for almost any quantity of interest, length, time, mass, energy, speed, rotation rate, acidity, concentration of this or that chemical in a solution, electric and magnetic field strengths, voltage, current, temperature, sizes of different types of living cells, sizes of organisms, chemical reaction rates, even of the same type of reaction, and so on. Rusting of a metal and an explosion are both burning, oxidation reactions; imagine the difference in the rates of oxidation in the two cases.

The numbers we give for any quantity depend on the system of units. Conventional units are chosen from the human scale, like meters, seconds, grams etc. In these units one has very large as well as very small numbers. Our intermediate human scale units may differ by hundreds or thousands, like meters and kilometers or inches and miles, while the range of scales Nature comes up with is much much wider. So no matter what units we use there will be very large numbers and very small numbers. We express a very large number like 1 000 000 000....., say 27 zeros , as 10^{27}, and a very small number like 0.000 000 000 ....001, say 54 zeros after the decimal point, and 1 in the 55th digit, as 10^{-55}. In standard *scientific notation* a number is written with an integer between 1 and 9, a decimal point followed by as many digits as needed for the *degree of accuracy* for that particular number, times 10 to a positive or negative power. Thus 123.456 is 1.23456×10^2, 0.000 000 000 123 is 1.23×10^{-10}, $-$ 10101.02 is $-\ 1.010102 \times 10^4$ etc. The number of powers of 10 is *the order of magnitude*,[1] and the number of digits before and after the decimal point is *the number of significant figures*.

Questions:

- What is the origin and definition of our human scale units, the meter, gram and second?
- Why do we have numbers like 12 and 60 in our system of time units?
- Why do we use the decimal system, expressing numbers in terms of powers of ten?

Among real numbers, as mathematical entities, only *rational numbers* can be expressed exactly, though this may require a very large number of digits. *Irrational numbers* like $\sqrt{2}$ and *transcendental numbers* like π require an infinite number of digits to be represented exactly. The number of significant figures used depends on the accuracy needed in the calculation.

Numbers used in science refer to measured quantities. Scientific numbers usually have *units* and *dimensions*. These numbers are meaningless without units. The accuracy of a number in science depends fundamentally on the accuracy of the measurements of the quantities making up that number. One may decide to work with less accuracy than the accuracy of the measurements, depending on the needs of the particular application. A

[1]For a scan through the different scales of size in nature, check the link to Powers of Ten on SUCourse. You can surf through the following internet sites, among others:
`http://microcosm.web.cern.ch/microcosm/p10/english/welcome.html`,
`http://micro.magnet.fsu.edu/primer/java/scienceopticsu/powersof10/`,
`http://www.wordwizz.com/pwrsof10.htm`,
`http://powersof10.com/` ;
for the original video: `www.kottke.org/06/06/powers-of-ten`.

measured quantity is quoted with errors of measurement, like for example $1.23 \pm 0.02 \times 10^{-6}$ m.

When one makes a calculation involving many numbers, each with its errors, the result has an error that can be calculated from the errors of the input numbers. When you do experiments in the lab, you will learn to estimate measurement errors and to make calculations of errors in derived quantities. When errors are not quoted, the number of significant places reflects the accuracy of a quantity.

In any calculation, the errors in a calculated result cannot be less than the errors of the least accurate input quantity. When quantities are added or subtracted, the number of decimal places in the answer is equal to the number of decimal places in the quantity with the smallest number of decimal places. When quantities are multiplied or divided, the number of significant figures in the answer is equal to the number of significant figures in the quantity with the smallest number of significant figures.

Question: Divide 4.25×10^{15} by 1.1924×10^{-5} using a pocket calculator. What does the calculator give? What is the answer in scientific notation expressed with the right accuracy?

Answer: $4.25/1.1924 = 3.56424....$ However, since the first number (4.25) has 2 decimal point accuracy, the answer should be accurate also up to 2 decimal points: 3.56. Therefore, the answer with the right accuracy would be 3.56×10^{20}.

Measurement Accuracy

Example 1: If you are getting weighed on an analogue bathroom scale with ticks in every kg, your measurement error is about 0.5 kg. You can then measure your weight with 0.5 kg increments: 56.4 ± 0.5 kg.

Example 2: If you are measuring the length of a ping-pong table (standard length 264 cm) with a 20-m measuring tape and a 20-cm ruler, what would your accuracy be in each case? Both the measuring tape and the ruler has ticks in every mm.
With the measuring tape, your measurement error would be about a few mm. So you can measure, for example, 263.8 ± 0.3 cm. If you are using the ruler though, you would need to make 13 or 14 consecutive measurements and each measurement would have a few mm of error. These errors would add up and your measurement accuracy would decrease. You can estimate the error to be 2 mm for one measurement, and if you make 14 measurements your total error would be $\sim$2.8 cm. So you could measure 265 ± 3 cm. The exact length of the table is within the range of both measurements but the accuracies are different.

4.3 Dimensional Analysis

Dimensional analysis is an important tool for checking calculations in science.

Quantities we use and their units are expressed in terms of mass M (kg), length L (meter, m) and time T (seconds, s), and charge Q (Coulombs, C). These units are the basic units of the Systéme International (SI).

The notation [ ] means "dimensions of". Here are the dimensions of velocity v, acceleration a and energy E:

$$[v] = L/T = M^0 L^1 T^{-1}$$
$$[a] = L/T^2 = M^0 L^1 T^{-2}$$
$$[E] = ML^2/T^2 = M^1 L^2 T^{-2}$$

Example: Time of Freefall

An apple falls from a tree branch of height h. Using dimensional analysis, we can find out how the time for the apple to reach the ground depends on other variables.

You may think, the freefall time t might depend on the initial height h, gravitational acceleration g, and maybe the mass of the apple m. So let's say:

$$t \propto h^\alpha \; g^\beta \; m^\gamma$$

This, in terms of their dimensions, becomes:

$$M^0 L^0 T^1 = [L]^\alpha \; [L/T^2]^\beta \; [M]^\gamma$$

The two sides of this equation must agree, so by looking at the powers of M, L, and T we can see that:

$$\begin{aligned} M &: 0 = \gamma \\ L &: 0 = \alpha + \beta \\ T &: 1 = -2\beta \qquad \rightarrow \beta = -1/2 \end{aligned}$$

Solving these equations, we find $\alpha = 1/2$, $\beta = -1/2$, and $\gamma = 0$. So:

$$\begin{aligned} t &\propto h^{1/2} \; g^{-1/2} \; m^0 \\ t &= C \sqrt{\frac{h}{g}} \; , \end{aligned}$$

where C is some constant.

4.4 Fundamental Constants

G is Newton's constant of gravity[2]. It has the value 6.67×10^{-11} m^3 kg^{-1} s^{-2}.

h is the Planck constant. It occurs in Quantum Mechanics, which is the subject of the last module of our course. The value is $h = 6.63 \times 10^{-34}$ Joule-sec. (The Joule (J) is the SI

[2]see the list of "Fundamental Physical Constants" in the front matter of this book

unit of energy, 1 J = 1 kg m^2 s^{-2}.) The Planck constant is usually used in the combination $\hbar \equiv h/2\pi$, called *"h-bar"*.

c is the velocity of light. Its value is $c = 3 \times 10^8$ m s^{-1} .

Other fundamental constants are the masses of the fundamental particles and the quantum of charge $e = 1.6 \times 10^{-19}$ Coulomb. The charge of the proton is e, and the charge of the electron is $-\,e$.

There are certain dimensionless combinations of fundamental constants. These are just numbers, without units, but they are numbers that tell us something important about the physical world.

An important example is the *"fine structure constant"*. The electrostatic *force* between an electron and a proton has the magnitude Ke^2/r^2. This depends on the distance r between the two particles but the strength of the electromagnetic interactions is reflected in the *"coupling constant"* Ke^2. The numerical values of the constant K and of the fundamental charge e depend on the system of units. It turns out that the combination of constants

$$\alpha \equiv \frac{Ke^2}{\hbar c} = \frac{1}{137}, \tag{4.1}$$

called the *"fine structure constant"* is *dimensionless.* The fine structure constant is the dimensionless coupling constant of electromagnetic interactions. For comparison, the dimensionless coupling constant of the strong nuclear force has a value of the order of 1.

4.5 Vectors

Consider a point P in three dimensional space. Referred to an *orthogonal* coordinate system made of the mutually perpendicular x, y and z axes meeting at *the origin* $O(0,0,0)$, P has the coordinates (x,y,z).

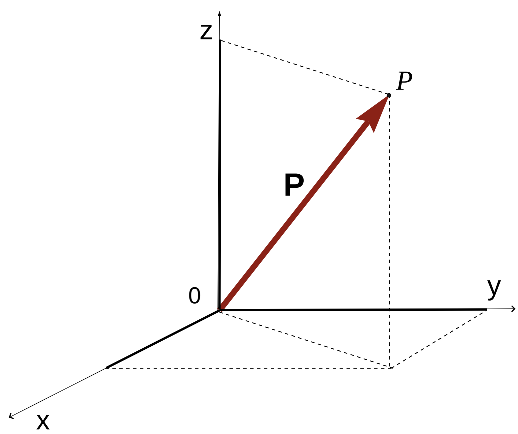


Figure 4.2:

The position of a particle at point P is completely specified by the numbers x,y and z, its coordinates in the given coordinate frame. The same information is depicted by an arrow starting from the origin of the coordinate frame, O with its tip at P. The set of the three numbers (x,y,z) or the equivalent graphic object, the directed line segment OP is called a *vector*. It is an object with a 'magnitude' and a direction. Vectors are denoted by a little arrow on top of their name ($\vec{P}$) or by **bold** type (**P**). Position, velocity acceleration, momentum, angular momentum, electric and magnetic fields are some of the vector quantities we will encounter in this course. A quantity without direction information is not a vector; it is called a scalar. Numbers or functions that refer to mass, time, speed, energy, work etc are scalars.

4.5.1 Magnitude

The *magnitude* of the vector P is the distance of the point P from the origin O. We denote the magnitude of P simply with P, without the bold type. The magnitude is a scalar, it is a nonnegative real number, given by

$$|\mathbf{P}| = P = (x^2 + y^2 + z^2)^{1/2}. \tag{4.2}$$

Where does this come from? Just the Pythagoras formula for finding the length of the long sideof a right angled triangle, as you can see from Figure 4.2.

There are many vectors whose magnitudes are the same as that of **P**: an infinite number of them – all points on the surface of the sphere with center O and radius P. **P** is unique among them all because its coordinates (x, y, z) specify a *direction* as well as a magnitude.

Questions:

- Can you think of vectors in two dimensions, in the given coordinate system, that have the same magnitude as **P**? Where are these vectors located?
- How about in one dimension?

4.5.2 Addition & Multiplication

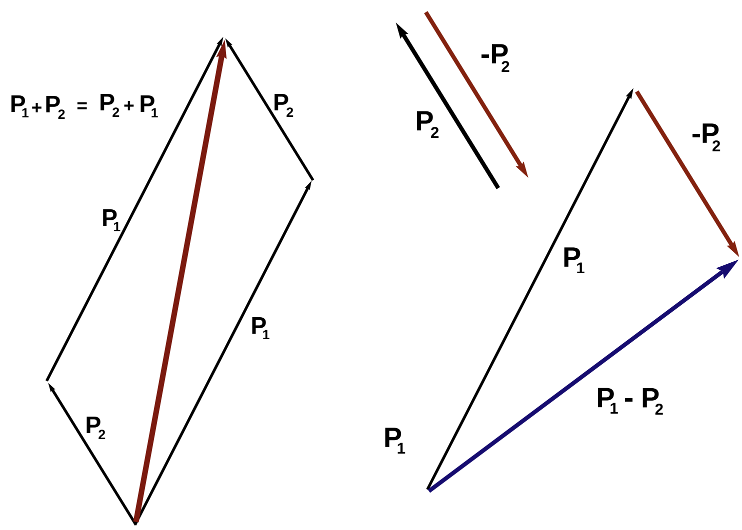


Figure 4.3:

Adding two vectors is an operation that is needed and useful in science. If $\mathbf{P_1} = (x_1, y_1, z_1)$ and $\mathbf{P_2} = (x_2, y_2, z_2)$ are two different displacements from the origin, going through the two of them, no matter in what order, brings us to the point with total displacement

$$\mathbf{P_1} + \mathbf{P_2} = \mathbf{P_2} + \mathbf{P_1} = (x_1 + x_2, y_1 + y_2, z_1 + z_2) \tag{4.3}$$

Geometrically the total ("*resultant*") vector is the diagonal of the parallelogram formed by $\mathbf{P_1}$ and $\mathbf{P_2}$ (see Figure 4.3).

Multiplying a vector by a number is simple: $a\mathbf{P} = (ax, ay, az)$.

> For example, 3.29 times $\mathbf{P}$ is a vector that is in the same direction as $\mathbf{P}$ but 3.29 times longer.
> $-\mathbf{P} = (-1)\mathbf{P}$ is a vector of the same magnitude but in the opposite direction as $\mathbf{P}$.

4.5.3 Unit Vectors

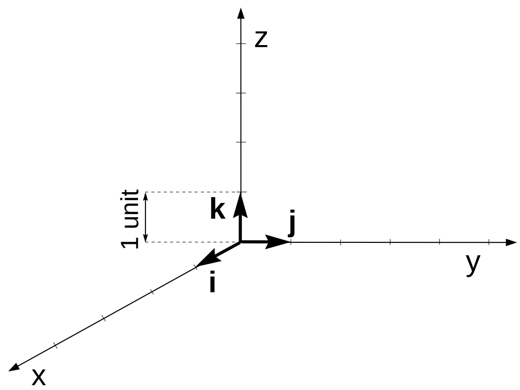


Figure 4.4:

One can write $\mathbf{P}$ as

$$\mathbf{P} = (x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \tag{4.4}$$

where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are *unit vectors*. These unit vectors are vectors of magnitude (or length) one unit along the x, y and z axes, respectively. The coordinates x, y, z are the *components* or *projections* of $\mathbf{P}$ along the x, y, z axes (See figure 4.4).

Solved Problem: Finding a Unit Vector

Find a unit vector in the direction of a vector $\mathbf{A} = 2\sqrt{2}\mathbf{i} - 5\mathbf{j} + 4\mathbf{k}$.

A unit vector has a magnitude of unity (= 1). So, we can define a unit vector in the direction of any vector by: $\hat{\mathbf{A}} = \mathbf{A}/|\mathbf{A}|$. Let's first find the magnitude of the vector $\mathbf{A}$:

$$\begin{aligned}|\mathbf{A}| &= ((2\sqrt{2})^2 + (-5)^2 + 4^2)^{1/2} \\ &= (8 + 25 + 16)^{1/2} \\ &= 7\end{aligned}$$

So, the unit vector is: $\hat{\mathbf{A}} = \dfrac{\mathbf{A}}{|\mathbf{A}|} = \dfrac{2\sqrt{2}}{7}\mathbf{i} - \dfrac{5}{7}\mathbf{j} + \dfrac{4}{7}\mathbf{k}$.

We can confirm that the magnitude of $\hat{\mathbf{A}}$ is 1, and it is in the same direction as the original vector $\mathbf{A}$.

4.5.4 Inner product

The "*inner product*" or "*scalar product*" or "*dot product*" of P_1 and P_2, denoted $\mathbf{P_1} \cdot \mathbf{P_2}$ is:

$$\mathbf{P_1} \cdot \mathbf{P_2} = \mathbf{P_2} \cdot \mathbf{P_1} \equiv x_1x_2 + y_1y_2 + z_1z_2 \tag{4.5}$$

The result of this product of two vectors is an ordinary positive, zero or negative real number, a "*scalar*", not a vector. This inner product turns out to be the product of the magnitudes

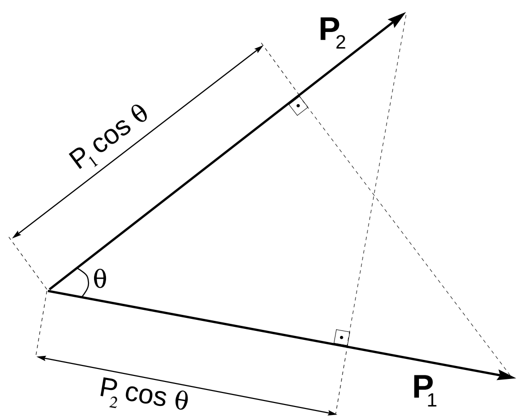


Figure 4.5:

of the two vectors times the *cosine* of the angle between them.

$$\mathbf{P_1} \cdot \mathbf{P_2} = P_1P_2\cos\theta \tag{4.6}$$

It is equal to P_1 times the length of the projection of $\mathbf{P_2}$ on $\mathbf{P_1}$ or P_2 times the length of the projection of $\mathbf{P_1}$ on $\mathbf{P_2}$ (see Figure 4.5).

Question: What is the value of $\mathbf{P_1} \cdot \mathbf{P_2}$ if $\mathbf{P_1}$ and $\mathbf{P_2}$ are perpendicular to each other? If they are parallel? Anti-parallel?

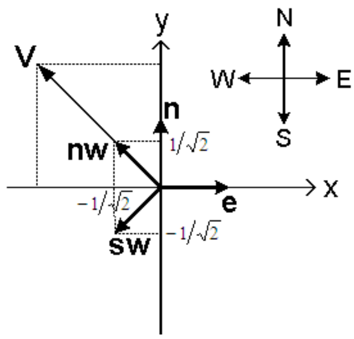
The inner product of a vector with itself defines the square of the length of the vector:

$$\mathbf{P} \cdot \mathbf{P} = \mathbf{P^2} = x^2 + y^2 + z^2 \tag{4.7}$$

Solved Problem: Inner Product

Inner product (or dot product) is useful when we would like to figure out the projection of a vector on a particular direction.
A car is moving North-West with a constant velocity of 50 km/h.

1. How can you define the unit vectors in the directions of North, East, North-West, and South-West, in terms of **i** and **j**? Take +y direction to be the North.



The unit vectors **n**, **e**, **nw**, and **sw** are shown in the figure. Since the unit vectors must have a magnitude of 1, using the figure we find:

$$\mathbf{n} = 0\mathbf{i} + 1\mathbf{j}, \qquad \mathbf{e} = 1\mathbf{i} + 0\mathbf{j}$$

$$\begin{aligned}\mathbf{nw} &= -(1)\cos 45^\circ \mathbf{i} + (1)\cos 45^\circ \mathbf{j} \\ &= -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} \\ \mathbf{sw} &= -(1)\cos 45^\circ \mathbf{i} - (1)\cos 45^\circ \mathbf{j} \\ &= -\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}\end{aligned}$$

2. What is the component of the velocity in the North and South-West directions?

Continues on next page...

...Continued from previous page

Using Equation 4.6,

$$\mathrm{v_N} = \mathbf{v} \cdot \mathbf{n} = (50)(1)\cos 45^\circ = 25\sqrt{2}\ km/h$$
$$\mathrm{v_{SW}} = \mathbf{v} \cdot \mathbf{sw} = (50)(1)\cos 90^\circ = 0$$

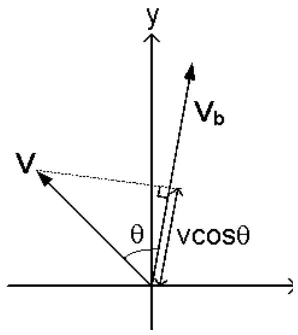
Alternatively, since we can write $\mathbf{v}$ as $\mathbf{v} = (-25\sqrt{2}\mathbf{i} + 25\sqrt{2}\mathbf{j})$ km/h, we can use Equation 4.5 to find the components:

$$\mathrm{v_N} = \mathbf{v} \cdot \mathbf{n} = [-25\sqrt{2}\mathbf{i} + 25\sqrt{2}\mathbf{j}] \cdot [0\mathbf{i} + 1\mathbf{j}] = 25\sqrt{2}\ km/h$$
$$\mathrm{v_{SW}} = \mathbf{v} \cdot \mathbf{sw} = [-25\sqrt{2}\mathbf{i} + 25\sqrt{2}\mathbf{j}] \cdot [-\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}] = 0$$

Here, it is easier to use the first method since the magnitudes of the vectors are given and the angles between the vectors can be easily found geometrically from the figure.

Using the second method, we can always find a projected component of any vector on any direction, even when the angle between the two vectors is not known.

3. A bus is going in another direction with a velocity $\mathbf{v}_\mathrm{b} = (10\mathbf{i} + 70\mathbf{j})$ km/h. What is the component of $\mathbf{v}$ in the direction of bus' motion?



First we find the unit vector in the direction of $\mathrm{v_b}$:

$$\mathbf{\hat{v}_b} = \frac{\mathbf{v}_\mathrm{b}}{|\mathbf{v}_\mathrm{b}|} = \frac{10\mathbf{i} + 70\mathbf{j}}{(\sqrt{10^2 + 70^2})} = \frac{\mathbf{i} + \mathbf{7j}}{5\sqrt{2}}$$

The component of $\mathbf{v}$ in the direction of $\hat{\mathrm{v}}_\mathrm{b}$ is

$$\mathbf{v} \cdot \mathbf{\hat{v}_b} = \frac{[-25\sqrt{2}\mathbf{i} + 25\sqrt{2}\mathbf{j}] \cdot [\mathbf{i} + 7\mathbf{j}]}{(5\sqrt{2})}$$
$$= 30\ km/h$$

We can also find the angle θ from this:

$$\theta = \cos^{-1}(\mathrm{v_B}/\mathrm{v}) = \cos^{-1}(30/50) = 0.93\ rad = 53.1^\circ$$

4.5.5 Vector product

In three dimensions, there is another product of two vectors called the "*vector product*", or "*outer product*" or "*cross product*":

$$\mathbf{P_1} \times \mathbf{P_2} = (y_1 z_2 - z_1 y_2,\ z_1 x_2 - x_1 z_2,\ x_1 y_2 - y_1 x_2) \tag{4.8}$$

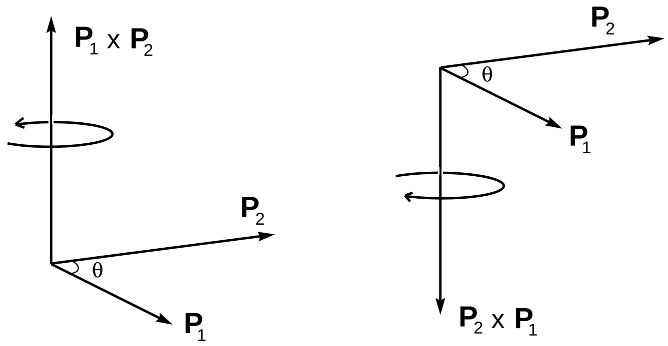


Figure 4.6:

Unlike the inner product, *the result of the cross product is itself a vector*. The magnitude of this product is P_1 times P_2 times the *sine* of the angle between them.

$$|\mathbf{P_1} \times \mathbf{P_2}| = P_1 P_2 \sin\theta \tag{4.9}$$

This means the magnitude of the cross product is P_1 times *the component of* P_2 *perpendicular to* $\mathbf{P_1}$, or the other way around.

The direction of $\mathbf{P_1} \times \mathbf{P_2}$ is perpendicular to the plane of $\mathbf{P_1}$ and $\mathbf{P_2}$, pointing in the direction specified by the right hand rule.

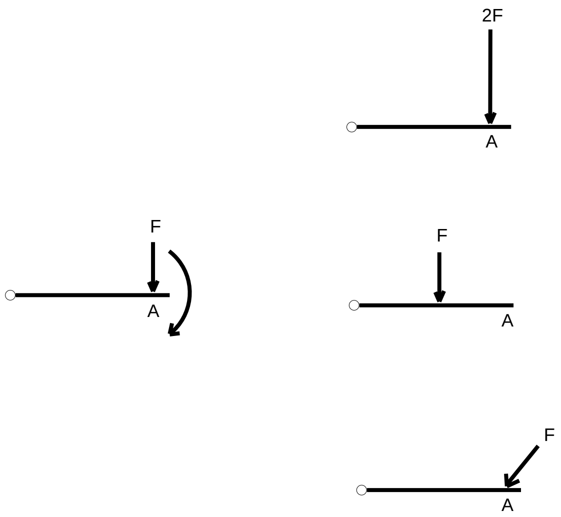
Question: Is $\mathbf{P_2} \times \mathbf{P_1}$ equal to $\mathbf{P_1} \times \mathbf{P_2}$? How are they related?

4.5.6 Why are vectors useful?

Many quantities in science are vectors. Examples are displacement, velocity, acceleration, force, electric field, magnetic field etc. You will learn about each of these quantities later in the course. Here we use these quantities as examples for vector concepts. Some important vector quantities are defined in terms of other vectors, with vector relations.

Example: Vector (Cross) Products

• **Torque:** $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$
When you open a door by pushing it, what determines how easy it is to open it? It depends on where you apply the force, how big the force is, and the direction of the force. Look at the below picture:



In the picture, a force F is applied at a point A. The door rotates in the direction shown. The pictures on the right show 3 different situations:

1. 2F is applied at A,
2. F is applied at a point closer to the hinge, and
3. F is applied at A with an angle.

In each case, would the rotating effect be larger than, smaller than, or the same as the case in the left picture? From our experience, we know that the rotating effect is larger when the moment arm (the distance from the hinge to where the force is applied) is larger, when the force is larger, and when the force is applied perpendicularly to the moment arm.

The quantity that puts the objects into rotation is called the ***torque***. Torque is defined as: $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$. The cross product here includes all the information we have discussed above. Torque (that is the ability to rotate the door) is largest when the applied force is the largest, when the moment arm is the largest, and when the angle between the applied force and the moment arm is 90°. Only the component of the force perpendicular to the moment arm has an effect on rotation.

Continues on next page...

...Continued from previous page

• **Lorentz Force: $\mathbf{F} = q\,\mathbf{v} \times \mathbf{B}$**
When a particle of charge q moves with velocity vector $\mathbf{v}$ through a magnetic field $\mathbf{B}$, the field exerts a force

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B} \tag{4.10}$$

on this particle. The force is proportional to the amount of charge on the particle (q), the component of the velocity of the particle perpendicular to the magnetic field ($\mathrm{v}_\perp$), and the magnetic field strength (or alternatively, the component of the magnetic field perpendicular to the direction of motion of the particle ($\mathrm{B}_\perp$) and the speed of the particle.) What is the direction of the force?

A vector may change with time. This means its magnitude and/or direction change. When calculating the rate of change of the vector (its 'derivative') with respect to time, changes in direction as well as changes in magnitude are taken into account. In circular motion, for example swirling a ball tied to the end of a string, the displacement vector keeps changing its direction all the time while the magnitude remains constant – the magnitude is the length of the rope.

Question: What is happening to the velocity vector in uniform circular motion?

In many problems in science, one is interested in the values of some vector at many different points in space. For example in studying the flow of water in a river, in principle we have to consider the velocity of water at all points in the river. This is a *velocity field*: we have values of the velocity $\mathbf{v}$ varying from point to point as a function of the coordinates (x, y, z) (or of $\mathbf{r} = (x, y, z)$, itself a vector). Thus $\mathbf{v} = \mathbf{v}(\mathbf{r}, t)$. Because of the scattering of light from the surface, and all the bubbles and white water and the leaves and other objects flowing along, we can actually see the velocity field on the surface of the river. If you take a snapshot you have a map of the velocity field at some moment in time. With this map you can study how the velocity vector varies from point to point. So it is meaningful to talk about the spatial variations (derivatives) of a vector quantity. In fact there are two different kinds of spatial derivatives of a vector field. We will not discuss derivatives of vector fields in this course. The following set of questions will lead you to a qualitative picture.

Questions: Make some maps, pictures of velocity fields, by drawing flow lines with arrows on them to picture the way water is flowing. Can you give examples of such maps in paintings, sculpture, other artwork?
Imagine your map is on some kind of plastic material that you can bend and stretch to straighten out your flow lines. There can be two types of features in your maps that you cannot get rid of by stretching or bending or compressing or otherwise distorting the map. What are these features?
These features determine the fundamental structure of the map, and are related to the two kinds of spatial derivatives of the vector field.

CHAPTER 4 - PROBLEMS:

1. The gravitational force between two protons has the magnitude $F_G = G\, m_p^2/r^2$, where G is the gravitational constant, m_p is the proton mass and r is the distance between the protons. Define and calculate a dimensionless gravitational coupling constant, in analogy with the fine structure constant. Look up the values of all the constants entering your calculation. Use scientific notation and work to 3 significant figure accuracy. Compare your result with the fine structure constant above:

   Is gravitation stronger or weaker compared to electromagnetism?

2. Problems on the Planck scale (C. Saçlıoğlu):

   The purpose of the following problems is to use dimensional analysis for deriving a basic mass, the Planck Mass m_{PL} , a basic timescale, the Planck time t_{PL}, and a basic length scale l_{PL} in terms of the three fundamental constants G, $\hbar$ and c.

   (a) What are the dimensions of G, $\hbar$ and c?

   (b) $m_{PL} = G^x \hbar^y c^z$. Find the powers x, y, z using dimensional analysis. To do this, you substitute the dimensions of G,$\hbar$, and c as powers of M, L and T and require that the overall dimension is like the left hand side, M.

   (c) Calculate the Planck mass m_{PL} in kilograms. Compare it with the proton mass.

   (d) Derive t_{PL}, in terms of powers of G, $\hbar$ and c, and calculate its value in seconds.

   (e) Derive l_{PL} in terms of powers of G, $\hbar$ and c, and calculate its value in meters.

   What you have derived are the mass, length and time scales at which gravitation and quantum mechanics are both important. These scales are relevant to the earliest times after the Big Bang, or in Elementary Particle Physics. Although we can calculate the Planck scales, understanding what happens at those scales requires the unification of gravitation (general relativity) and quantum mechanics. This is still an unsolved fundamental problem.

3. Newton's Law of Gravitation gives the force F between two particles of masses m_1 and m_2 and separation r as:

$$F = \frac{Gm_1m_2}{r^2}$$

   where G is the universal gravitational constant. Find the dimension of G in terms of mass, length and time, and give its units in the SI system.

4. Hooke's Law: A mass is attached to a spring. The spring exerts a force that is directly proportional to the distance the mass is pulled or pushed from the equilibrium position: $F = -kx$. Find the dimensions of the spring constant k in terms of mass, length and time, and its units in the SI system.

5. The length of the Great Wall of China is 6700 km. Estimate roughly the area of China. Assume that the remaining borders and the sea coast of China have the same total length as the Great Wall.

6. The magnitude of the electric force that a charge q_1 applies on a charge q_2 at distance d is:

$$F = \frac{Kq_1q_2}{d^2}$$

   In the SI system the constant K is usually expressed in terms of another constant, ϵ_0 as $K = 1/(4\pi\epsilon_0)$. Find the dimensions and units of K and ϵ_0.

7. The annual rain and snowfall over an area of 2.5×10^5 km^2 in Eastern Turkey is 500 kg/m^2 . If this flows into 10 rivers in equal quantities throughout the year, what is the *flow rate* in each river in units of tons/s? If a river has a depth of 1 m and a width of 50 m under a certain bridge, what is the amount of water that passes below the bridge every second, in units of $kg/m^2/s$? [Note: This is called *flux*]

8. Take the following two vectors:

$$\mathbf{P_1} = (-2, 1, 2) \qquad\qquad \mathbf{P_2} = (1, -2, 2)$$

   Evaluate:

   (a) $\mathbf{P_1} \cdot \mathbf{P_2}$
   (b) $|\mathbf{P_1}|$ and $|\mathbf{P_1}|$
   (c) The angle between $\mathbf{P_1}$ and $\mathbf{P_2}$
   (d) $\mathbf{P_1} \times \mathbf{P_2}$
   (e) $\mathbf{P_2} \times \mathbf{P_1}$

9. Repeat the above problem (Problem 8) with:

$$\mathbf{V_1} = (2, 2, 1) \qquad\qquad \mathbf{V_2} = (1, 2, 1)$$

10. A proton with charge $q = 1.6 \times 10^{-19}$ *Coulomb*, velocity $\mathbf{v} = 3 \times 10^5$ m/s $\mathbf{i}$, enters an area with magnetic field $\mathbf{B} = 10^8$ $Tesla$ $(\mathbf{i} + \mathbf{j} + \mathbf{k})$.

    (a) What is the force $\mathbf{F}$ on this proton? [Recall Equation 19.7]
    (b) What is the magnitude of this force ?

    Note: The Tesla and the Coulomb are correct SI units, so you do not have to worry about unit conversions. The resulting force will be in the correct SI units of Newtons (N).

11. Convert the following quantities to the new units given. Write your answers using scientific notation (e.g., 5.1×10^3).

    (a) 43.7 cm/s to m/s
    (b) 2408 g cm/s^2 to Newtons (= kg m/s^2)
    (c) 1397 kg cm^2/s^2 to Joules (= kg m^2/s^2)

12. A plane is flying with a constant velocity of $\mathbf{v}_\text{p} = 100\mathbf{i}$ km/h, according to the plane's speedometer and gyroscope. You are observing the plane from a stationary point on the ground.

What is the plane's velocity relative to you on the ground (magnitude and direction in km/h), when a wind is blowing at a constant velocity $\mathbf{v}_{\rm w}$ in each of the following cases?

(a) $\mathbf{v}_{\rm w} = 20\mathbf{i}\,\text{km/h}$

(b) $\mathbf{v}_{\rm w} = -20\mathbf{j}\,\text{km/h}$

(c) $\mathbf{v}_{\rm w} = 10\ \mathbf{i} + 5\ \mathbf{j}\,\text{km/h}$

Chapter 5

How do we describe motion?

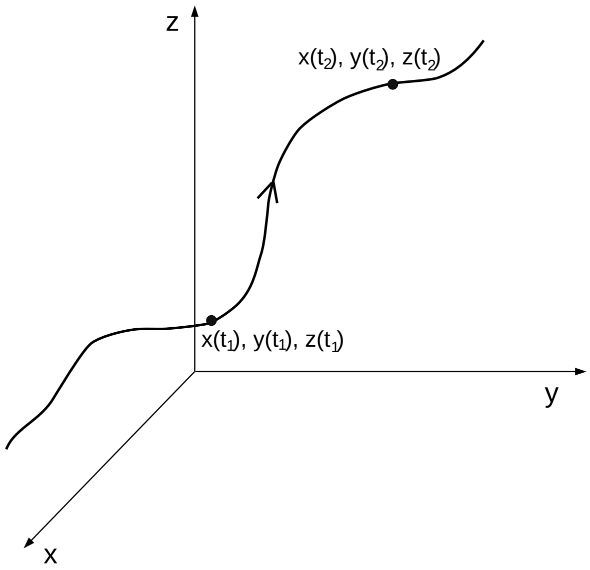


Figure 5.1: Trajectory of an object in three dimensional space.

For objects in our daily experience and, based on that experience, in the realm of classical physics, one notes where an object is at each moment in time. If the object is "small enough" to be described as a point, its position is given by its coordinates in three dimensional space, (x, y, z). These coordinates are some numbers referring to a coordinate system. As time goes on, the object (a "point particle") will be at different points (x_1, y_1, z_1) at t_1, then (x_2, y_2, z_2) at t_2, then (x_3, y_3, z_3) at t_3 and so on. We can measure all these and note the motion, the trajectory, as a set of points with coordinates $x(t_i)$, $y(t_i)$, $z(t_i)$ corresponding to the times t_i at which we made the measurements (see Figure 5.1). In principle these measurements can be made as frequently as one wants, and represent the motion of the particle. The functions $x(t)$, $y(t)$, $z(t)$ give the *trajectory* of the particle. The numerical values of $x(t)$, $y(t)$, $z(t)$ will depend on the coordinate system. Plotting the functions $x(t)$, $y(t)$ and $z(t)$ in three dimensional space will give the actual path of the object which is some kind of simple or complicated curve. This curve is the actual trajectory of the particle which of course does not depend on the coordinate system.

Questions:

- Is the object "small enough" to be described by a point?
- What does "small enough" mean?
- If the object is not small how do you describe its motion?
- How do you divide an object up into parts?
- How about the case of a rigid body? A fluid?

The particle moves along, passing through all the points on its trajectory. Motion means something more: *We can measure how fast it is moving.* On the same trajectory there are an infinite number of imaginable motions each characterized by a different sequence of speeds as a function of time.

The point of *kinematics* is describing *how* the particle moves. We will first build the language to describe motion. We will pick up the necessary mathematics, *elementary Calculus.* Then we move on to the *why*s: how *forces* determine the motion (*dynamics*) and how things affect each other: what are the forces?

We will consider the motion of a particle in one dimension. One dimension is simple, and it is enough: We will learn how to handle the one dimensional motion $x(t)$. Then the most complicated three dimensional motion is more of the same: $y(t)$ and $z(t)$ are handled in the same way as $x(t)$.

5.1 Average velocity

The particle is at point $P_1 = (x_1, y_1, z_1)$ at time t_1 and at point $P_2 = (x_2, y_2, z_2)$ at time t_2. If we know all this we know that the average speed of the particle between these two points was $(x_2 - x_1)/(t_2 - t_1)$ in the x direction, and similarly in the y and z directions.

$$v_x = \frac{(x_2 - x_1)}{(t_2 - t_1)} \qquad v_y = \frac{(y_2 - y_1)}{(t_2 - t_1)} \qquad v_z = \frac{(z_2 - z_1)}{(t_2 - t_1)} \tag{5.1}$$

We have measured the (average) velocity vector $\mathbf{v} = (v_x, v_y, v_z)$.

Classical Mechanics vs. Quantum Mechanics

In our daily experience and in Classical Mechanics we can measure both $\mathbf{x}$ and $\mathbf{v}$ together all the time. Not so in Quantum Mechanics! At small scales, as in atomic physics, and in general in properties of matter arising from atomic properties, experiments show that "particles" are also "waves". It is not possible to describe their motion in terms of a precise position and a precise velocity at the same time. We will discuss this later when we study quantum mechanics and the structure of the atom. For the time being we return to the familiar concepts of point particle, position, velocity, noting that these concepts are valid and applicable in a wide range of situations, to great accuracy, but are not applicable in certain very important and interesting realms of nature.

Our particle is moving faster or slower, maybe back and forth in the x direction: its x coordinate is changing all the time. The average speed between times t_1 and t_2 is

$$v = \frac{x(t_2) - x(t_1)}{t_2 - t_1} \tag{5.2}$$

But between the times t_1 and t_2 the particle is sometimes faster, sometimes slower. To get at its velocity at time t we must compare its position at time t with its position only a very very very short interval Δt later, so that the two positions are *almost* instantaneous; they both measure the positions almost at the same time. So we get a much better estimate of the velocity at time t:

$$v = \frac{x(t + \Delta t) - x(t)}{\Delta t} = \frac{\Delta x}{\Delta t} \tag{5.3}$$

$\Delta x = x(t + \Delta t) - x(t)$ is the *little bit of distance* the particle travels during the short time interval Δt. This $v = \Delta x/\Delta t$ is still an average velocity but over a short time interval. For example t might be 10 s, and Δt might be 1 s. This means we measure the particle's position at 10 s, and then again at 11 s. The distance gone between 10 s and 11 s divided by the time spent, 1 s, gives the average velocity of the particle between 10 and 11 s. Now if we did these measurements and calculation for $\Delta t = 0.01\ s$, we would obtain the average velocity between 10 s and 10.01 s . The shorter the interval Δt, the closer the result is to the instantaneous velocity at $t = 10\ s$. This business of *imagining the same calculation* of the velocity but over shorter and shorter time intervals to reach an actual value is the mathematicians' concept of "*limit*":

$$v(t) = \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} \tag{5.4}$$

In practice the shortest time interval in this measurement is the time taken to determine the position of the particle to the best available accuracy in the measurement. Comparing successive positions at short time intervals practical for the particular experiment is enough to determine the instantaneous velocity. We do not take limits experimentally but we can imagine them.

So, Δx is the small change in position accompanying the small change in time Δt. The ratio $\Delta x/\Delta t$ is called "the rate of change of x with respect to t". The velocity at time t is the instantaneous rate of change of x with respect to t, defined as "*the limit of* $\Delta x/\Delta t$ *for shorter and shorter* Δt". This limit process of calculating $\Delta x/\Delta t$ for shorter and shorter intervals is called *taking the derivative of x with respect to t.*

> The velocity of a particle is the derivative of its position with respect to time.
> There is a short notation for this:
> Instead of "$v(t) = \lim_{\Delta t \to 0} [\Delta x/\Delta t]$" one says "$v(t) = dx/dt$".

Solved Problem: Speed of a Shuttle

A trip from the gate of Sabancı University to the toll booths (Gişeler) towards İstanbul takes about 20 minutes with the shuttle bus. The total distance shown on the shuttle's kilometre counter is 25 km.

(a) What is the shuttle's average speed?

$$\text{Average speed is: } v = \frac{\Delta x}{\Delta t} = \frac{25\ km}{20\ min} = 1.25\ km/min$$

How much is this in SI units?

$$v = \frac{25\,\cancel{km}}{20\,\cancel{min}} = \frac{1000\ m}{1\,\cancel{km}} = \frac{1\,\cancel{min}}{60\ s} = 21\ m/s$$

(b) Is this a reasonable speed for TEM?

We usually talk about cars' speeds in km/h, so we need to convert 21 m/s to km/h to be able to tell if this is a reasonable speed for TEM:

$$v = 21\,\frac{\cancel{m}}{\cancel{s}}\,\frac{1\ km}{1000\,\cancel{m}}\,\frac{3600\,\cancel{s}}{1\ h} = 76\ km/h$$

This is a bit slow for TEM, where the speed limit is 120 km/h. Note, however, that this is an *average speed* for the entire trip. Around the Sabancıcampus, the shuttle cannot go that fast.

(c) How would you figure out the average speed of the shuttle on TEM?

You can consider only the part of the trip when the shuttle is on TEM. If you take recordings of time and distance of this part of the trip, you can find an average speed that is much larger and closer to the values that the driver sees on the speedometer.

(d) Can you think of a practical way to calculate the speed of the shuttle passing by a certain point on TEM more accurately? (Not by using the method of asking the driver)

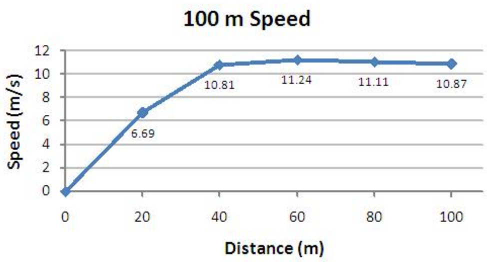
Use anything that you know the distance of: roadside posts indicating distances, road signs (...m to Exit 128), a tunnel, bridge, large building along the TEM that you know the length of. Then measure the time (with your watch) for the shuttle to pass the distance.

Solved Problem: Speed of a Sprinter

Men's 100-m run world record is 9.58 s. What is the average speed of the sprinter in m/s?

$$v = \frac{100\ m}{9.58\ s} = 10.4\ m/s$$

This is only the *average speed* of the sprinter. The instantaneous speed at any point during the 100-m run can exceed or fall below this speed. The graph below shows actual measurements of instantaneous speeds taken at an Olympic final[1]:



We can see that the sprinter reached the fastest speed at around 60 m.
Let's compare this with a Cheetah. A Cheetah can run at 120 km/h – would the fastest man stand a chance if a cheetah were chasing him?

To compare, we need to convert the man's speed to km/h:

$$v = 10.4 \frac{\cancel{m}}{\cancel{s}} \frac{1\ km}{1000\ \cancel{m}} \frac{3600\ \cancel{s}}{1\ h} = 37.4\ km/h.....$$

Sorry, no chance...

5.2 How does one calculate dx/dt?

dt is a very short time interval: shorter than the shortest you can imagine, zero *in the limit*. But so is dx. In the shortest time interval the particle moves a tiny distance even if it is going very fast. So dx/dt is a small number divided by a small number; zero over zero in the limit. Its value can be calculated. It just depends on how dx is related to dt; how exactly the distance step becomes shorter as the time interval gets shorter. Well, all this information is in the function $x(t)$. If you can figure out where the particle is at any moment in time then you can calculate its velocity at any moment.

[1]adapted from `http://www.brianmac.co.uk/sprints/`

Example: A particle is moving along the x-axis. Let us call the starting point $x = 0$. The particle starts there at $t = 0$. You measure the position $x(t)$ at many later moments t, make a plot and find that the formula $x(t) = Ct^2$ describes the motion. C is some constant.
Make up a Table of x (in m) and t (in sec) values which illustrate the formula $x(t) = Ct^2$ with some numerical value of C that you choose.
Now let us calculate $v(t)$. First the average velocity between time t and time $t + \Delta t$:

$$x(t + \Delta t) = C(t + \Delta t)^2 = C(t^2 + 2t\Delta t + (\Delta t)^2) \tag{5.5}$$

$$\Delta x = x(t + \Delta t) - x(t) = C(t^2 + 2t\Delta t + (\Delta t)^2) - Ct^2 \tag{5.6}$$

$$\Delta x = C(2t\Delta t + (\Delta t)^2) \tag{5.7}$$

$$\frac{\Delta x}{\Delta t} = C(2t + \Delta t) \tag{5.8}$$

$$v(t) = \frac{dx}{dt} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = 2Ct \tag{5.9}$$

$v(t) = dx/dt = 2Ct$ because in the limit Δt is certainly negligible compared to $2t$!

So we know how to calculate $v(t)$ with our prescription. Are we going to go through all this every time we want to calculate a velocity?

No. There are rules for finding the derivative (this is also called *differentiating*) of all sorts of functions. These rules all stem from the basic definition of the derivative. Knowing the rules for differentiation for the most common and important (and usually simple) functions one can get results easily.

In the example above x was proportional to t^2, the 2nd power of t, and then v turned out to be proportional to 2 times t.

Some other rules for differentiation are given below. Derive these from the basic definition of dx/dt. Understand and use these rules; you will learn more about them in your Mathematics courses.

1. If $X(t) = Cx(t)$, C is a constant, then $dX/dt = Cdx/dt$.
2. $dC/dt = 0$.
3. If $x(t) = u(t) + s(t)$, like for example $x(t) = 3t^2 - 1/t$, then $dx/dt = du/dt + ds/dt$.
4. If $x(t) = u(t)s(t)$, for example $x(t) = t^{1/2}t^3$, then $dx/dt = u(t)\ ds/dt + du/dt\ s(t)$.
5. If x is given as a function of say y, $x = x(y)$, and y in turn depends on t, $y = y(t)$, such that $x = x(y(t))$, the derivative is calculated by using a simple *chain rule* :
$$\frac{dx}{dt} = \frac{dx}{dy} \cdot \frac{dy}{dt} \tag{5.10}$$
6.
$$\frac{dt}{dx} = \left(\frac{dx}{dt}\right)^{-1} \tag{5.11}$$

We have figured out how to calculate $v(t)$ from $x(t)$. We can of course now find out how the velocity is changing from moment to moment. The rate of change of velocity is called ***acceleration***:

$$a(t) = \frac{dv(t)}{dt} \tag{5.12}$$

Now let us go back to our example. For a motion through positions $x(t) = Ct^2$, the velocity varies as $v(t) = 2Ct$. So what is the acceleration?

$a(t) = dv/dt = 2C$ according to our rules of differentiation.

$x(t) = Ct^2$ corresponds to constant acceleration.

An example of motion with constant acceleration is the motion under the influence of gravity. Gravity near the surface of the Earth provides constant acceleration (to a good approximation, depending on how near one is to the surface: see Problem 1 in Chapter 2).

CHAPTER 5 - PROBLEMS:

1. You are at the Taksim Metro stop at $t = 0$. You pass Osmanbey, go to Şişli, and then come back to Osmanbey. The position *vs* time graph of your trip is shown in Figure 5.2.

   (a) What is the average speed of the train between Taksim and Şişli?

   (b) What is the average speed of the train during the entire trip between Taksim and Osmanbey (after Sişli)?

   (c) Sketch the train's velocity *vs* time graph

   (d) Sketch the train's acceleration *vs* time graph.

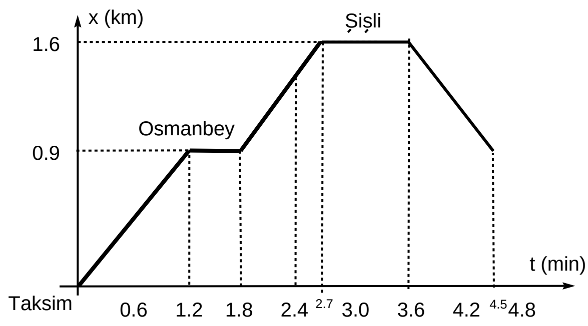


Figure 5.2: Problem 1

2. Show that if $x(t) = Ct^n$ then $v(t) = Cnt^{(n-1)}$.

3. A child throws her ball up vertically. The ball's height at every instant is described by the function:
$$h = 20t - 5t^2$$
h is in meters and t is in seconds.
   (a) What is the velocity $v(t)$ of the ball at time t?
   (b) What is the acceleration $a(t)$ of the ball at time t?
   (c) What is the height, velocity and acceleration at the times $1\ s,\ 2\ s,\ 3\ s$?

4. An object's motion can be divided into motion along the x direction and motion along the y direction as follows
$$x = 15t \qquad\qquad y = 20t - 5t^2$$
Calculate
$$\frac{dy}{dx} =?$$
Hint: Calculate $\frac{dy}{dx}$ in two ways:
   (a) by using the 'chain rule'
   (b) by substituting $t(x)$ into $y(t)$ first and then taking the derivative $\frac{dy}{dx}$.

5. Using the rules above evaluate dx/dt for
   (a) $x(t) = f(t) + g(t)$
   (b) $x(t) = f(t)g(t)$
   (c) $x(t) = f(g(t))$

   where $f(t) = t^{1/2}$ and $g(t) = 2t^3$.

6. I leave my house in the morning and walk to our bakkal to get the daily paper. The bakkal shop is 500 m north of my house. Then I go to the bus stop 200 m east of the bakkal. I meet with my friend Ayşe. We walk about 1 km southeast of the bus stop to our school.

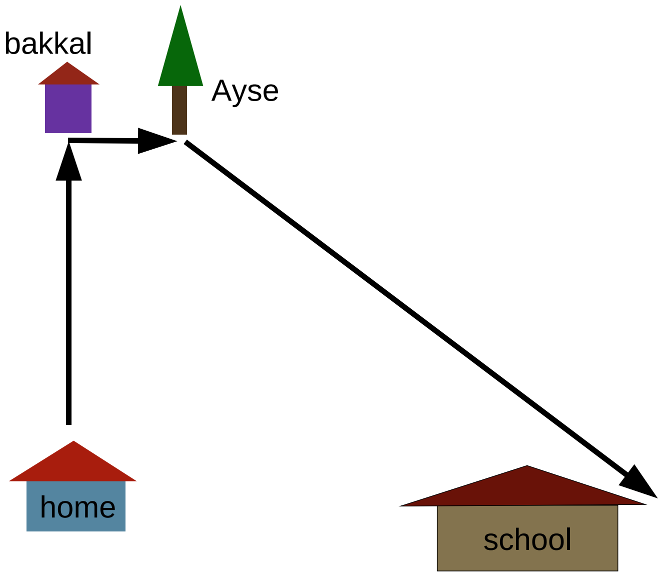


Figure 5.3:

(a) What is the actual distance that I walk from my house to school?

(b) If this trip takes about half an hour, what is my average speed?

(c) What is my average velocity vector?

7. A stone thrown at an angle of θ from the horizontal has the displacement vector:

$$\mathbf{r} = 5(\cos\theta)\, t\ \mathbf{i} + (5(\sin\theta)\, t - 5t^2)\ \mathbf{j} \quad \text{m}$$

(a) Find the velocity $\mathbf{v}(t)$

(b) Calculate the initial speed of the stone.

(c) Find the acceleration $\mathbf{a}(t)$

(d) Find the path of the stone $y(x)$

(e) Find the speed $v(t)$

8. Figure 5.4 shows 6 different position (x) vs. time (t) graphs. For each graph, sketch a velocity (v) vs. time (t) graph.

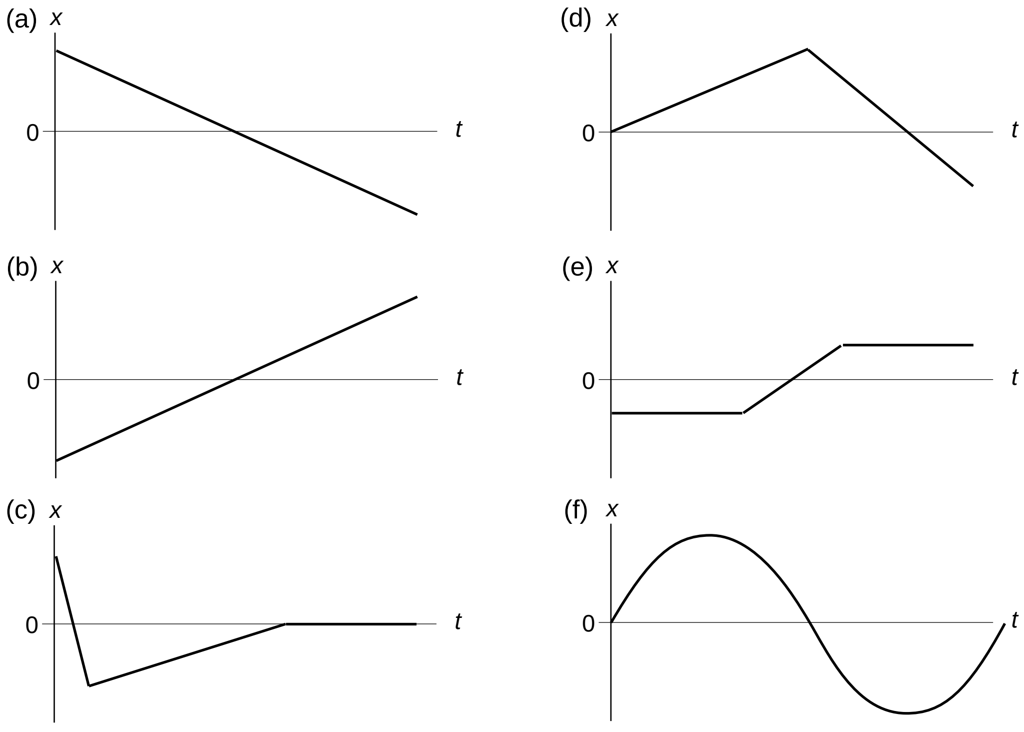


Figure 5.4:

Chapter 6

How to find out where the particle is?

Suppose a particle is at some point $x(0)$ at time $t = 0$, and it is moving with velocity $v(0)$. From that point on you have frequent observations of the velocity $v(t_n)$ at times $t_n = n\Delta t$.

Where is the particle at some final time t?

You are on a bike following a straight line. The bike is equipped with a speedometer. You record your speed at regular intervals Δt. You have no other means of figuring out where you are. (As a scientific experiment this is a pretty reckless design! You are doing it in your thoughts or your dreams. It is all for the sake of learning!). Can you figure out where you are?

Well, you started at the initial position $x(0)$. Then after a time interval Δt, at time t_1 you are at

$$x(t_1) = x(0) + v(0)\Delta t, \tag{6.1}$$

approximately: maybe during the interval Δt the speed changed from its initial value $v(0)$. This will be a good approximation if the interval Δt is short enough. Then from time t_1 to time t_2 you moved along at approximately the velocity $v(t_1)$. So at time t_2 you are at

$$x(t_2) = x(t_1) + v(t_1)\Delta t = x(0) + v(0)\Delta t + v(t_1)\Delta t \tag{6.2}$$

From time t_n to time t_{n+1} you move through an extra bit of distance $v(t_n)\Delta t$. So

$$x(t_{n+1}) = x(0) + v(0)\Delta t + v(t_1)\Delta t + v(t_2)\Delta t + \text{........} + v(t_n)\Delta t \tag{6.3}$$

If you want to find the total distance gone till time t, you measure the velocity at many times between 0 and t, let us say at regular intervals $\Delta t = t/N$. Then

$$x(t) - x(0) = \sum_{n=0}^{N-1} v(n\Delta t)\Delta t \tag{6.4}$$

The notation

$$\sum_{n=0}^{N-1} v(n\Delta t)\Delta t$$

means: "Sum of $v(n\Delta t)\Delta t$ for all values of n from $n = 0$ to $n = N - 1$".

In expanded form, Equation 6.4 is:

$$x(t) - x(0) = v(0)\Delta t + v(t_1)\Delta t + v(t_2)\Delta t + ... + v(t_N - 1)\Delta t$$

In other words, the total distance you went from time 0 to time t is the sum of the little steps you took in between. You have to calculate the distance in these little steps because the speed was different at each moment. This will become more and more accurate if you make your time intervals shorter and shorter. So:

$$x(t) - x(0) = \lim_{\Delta t \to 0} \sum_{n=0}^{N-1} v(n\Delta t)\Delta t \tag{6.5}$$

This kind of sum, evaluated in the limit of $\Delta t = 0$, is called an *integral.* There is a special notation:

$$x(t) - x(0) = \lim_{\Delta t \to 0} \sum_{n=0}^{N-1} v(n\Delta t)\Delta t = \int_0^t v(t')dt' \tag{6.6}$$

Now look at a graph of the velocity as a function of time (Fig 6.1). The position $x(t)$ is related to the $v(t)$ curve geometrically in a simple way:

$x(t) - x(0)$ is just the area under the $v(t)$ curve from 0 to t.

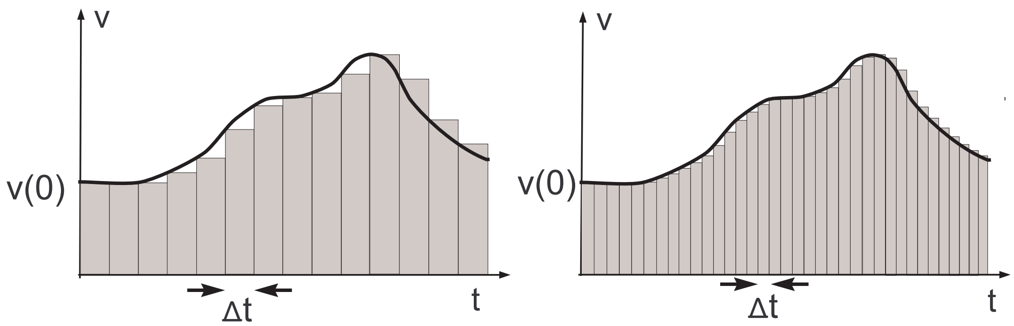


Figure 6.1: Graph of velocity as a function of time. The area under the curve $v(t)$ gives $x(t) - x(0)$. The time intervals Δt employed in the calculation of $x(t) - x(0)$ are longer in the case shown in the left panel. The right panel shows a more accurate calculation of $x(t) - x(0)$, using shorter Δt.

Question: How do you calculate $x(t) - x(0)$?

Well, $v(t) = dx/dt$. So, if you know $v(t)$ and want to find $x(t)$, your problem is the inverse of taking the derivative:

What function $x(t)$ has this given $v(t)$ as its derivative?

So you go and look at your table of functions and their derivatives, and look for your $v(t)$ in the "derivatives" column. Then go back and read the function $x(t)$ which has that derivative. Sometimes you find a known function in the table. (Sometimes you do not: nobody has done that integral before. Then you have to do it with mathematical tricks and numerical work, or with computer programs based on the above definition of the integral.)

Example: Suppose $v(t) = At^n$. In words, the velocity at time t is A times t to the power n. What is the position as a function of time?
The answer is

$$x(t) = A\frac{t^{n+1}}{(n+1)} + C$$

where C is a constant.

Why?

Because if I take the derivative of this $x(t)$, I get

$$v(t) = \frac{dx}{dt} = (n+1)A\frac{t^n}{(n+1)} + \frac{dC}{dt}$$

using the formula we derived before for the derivative of a power law t^n, and the other simple rules of differentiation. Now dC/dt is zero since C does not change, C is a constant. So the derivative of $x(t) = At^{n+1}/(n+1) + C$ is indeed $v(t) = At^n$. Note that the C that doesn't appear in $v(t)$ has to be there in $x(t)$. All the formulas for $x(t)$ with different constants C give the same $v(t)$.

If we want to find how far our particle went from time t_1 to time t_2 with this velocity law $v(t) = At^n$ we need to integrate $v(t)$ between times t_1 and t_2:

$$x(t_1) - x(t_2) = \int_{t_1}^{t_2} v(t)dt \tag{6.7}$$

$$\begin{aligned} x(t_1) - x(t_2) &= \int_{t_1}^{t_2} At^n dt \\ &= \left[\frac{At_2^{n+1}}{(n+1)} + C\right] - \left[\frac{At_1^{n+1}}{(n+1)} + C\right]. \end{aligned} \tag{6.8}$$

The constant C cancels out and there is no ambiguity.

$$x(t_1) - x(t_2) = \frac{At_2^{n+1} - At_1^{n+1}}{(n+1)}. \tag{6.9}$$

Solved Problem: Accelerating car

A race car accelerates to 100 km/h from rest in 2.5 s.
(a) What is the average acceleration (in SI units)?

First, we need to convert km/h to the SI units of velocity, m/s.

$$100\ \frac{km}{h} = 100\ \frac{\cancel{km}}{\cancel{h}}\ \frac{1000\ m}{1\ \cancel{km}}\ \frac{1\ \cancel{h}}{3600\ s} = 27.8\ m/s$$

So, the average acceleration is:

$$a = \frac{\Delta v}{\Delta t} = \frac{v_f - v_i}{\Delta t} = \frac{(27.8 - 0)\ m/s}{2.5\ s} = 11.1\ m/s^2$$

(b) If you consider the acceleration to be constant during this time, how much distance does the car travel in 2.5 s?

Here, we have $a(t) = 11.1\ m/s^2$, and the velocity function $v(t)$ can be found by:

$$v(t) = \int a(t)dt = (11.1t + \cancel{v_0}^{\,0})\ m/s.$$

Then, the distance function $x(t)$ is found by:

$$x(t) = \int v(t)dt = \int 11.1t\ dt = \frac{11.1t^2}{2} + \cancel{x_0}^{\,0}$$

Now we know the exact motion of the car in every instant of t, with:

$$x(t) = \frac{11.1t^2}{2}\ m$$
$$v(t) = 11.1t\ m/s$$

So, the position of the car at $t = 2.5\ s$ is

$$x(t = 2.5\ s) = \frac{11.1\ (2.5)^2}{2} = 34.7\ m$$

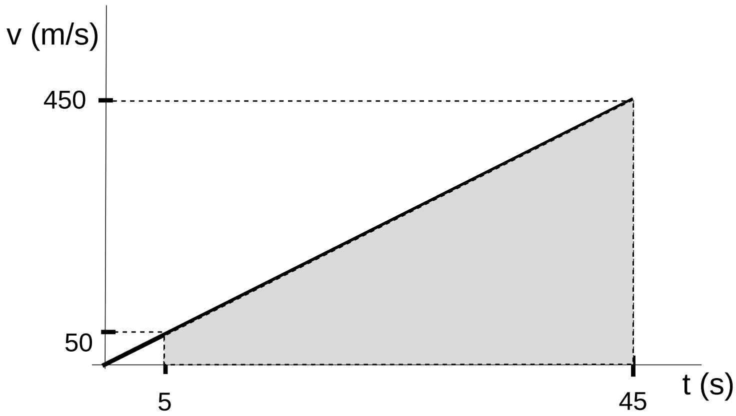
Since the car starts from $x = 0$ m, this is the distance the car traveled in 2.5 s.

CHAPTER 6 - PROBLEMS:

1. Suppose the velocity of an object changes in time as:

$$v(t) = 10\ t\ m/s.$$

How far does this object travel from the time $t_1 = 5\ s$ to $t_2 = 45\ s$? Find the answer



using the following methods:

(a) approximately, by using Equation 6.4 from t_1 to t_2 with $\Delta t = 5\ s$.

(b) to a better approximation, with $\Delta t = 1\ s$.

(c) exactly, by evaluating the integral

$$\int_{t_1}^{t_2} v(t)dt.$$

(d) exactly, by calculating the area under the $v(t)$ graph, between $t = 5\ s$ and $t = 45\ s$.

2. An object is thrown vertically upwards from an initial position $z(0) = 3\ m$, with an initial velocity $v_z(0) = 10\ m/s$. The acceleration due to gravity is $a_z(t) = g \cong -10\ m/s^2$. Use integration to find:

   (a) the velocity $v_z(t)$ at any later time t.

   (b) the height $z(t)$ at any later time t.

   (c) Where is the object and what is its velocity at time $t = 1\ s$? At $t = 2\ s$?

3. For vertical motion starting from $z(0) = 0$, with initial velocity $v_z(0) = 100\ m/s$, take $a_z(t) = g \cong -10\ m/s^2$, and use integration to find:

   (a) the velocity $v_z(t)$ at any later time t,

   (b) the height $z(t)$ at any later time t.

   (c) Graph the velocity as a function of time from $t = 0$ to $t = 20\ s$.

   (d) Use the graphical property of integration, that $z(t)$, the integral of $v_z(t)$, is given by the area under the $v_z(t)$ graph, to find the height $z(t)$ at $t = 5,\ 10,\ 15,\ 20\ s$.

4. Any object near the surface of the Earth is accelerated downward with the approximately constant gravitational acceleration $\mathbf{a}(t) = -g\mathbf{k}$. A projectile starts motion from initial position $\mathbf{r}(0) = h\mathbf{k}$ with initial velocity $\mathbf{v}(0) = v_x(0)\mathbf{i} + v_z(0)\mathbf{k}$. $\mathbf{i}$ is the unit vector in the horizontal x direction, and $\mathbf{k}$ is the unit vector in the upward vertical z direction.

   (a) Find the vertical velocity $v_z(t)$ by integrating the acceleration.

(b) Find the height $z(t)$ by integrating the vertical velocity $v_z(t)$.

(c) Find the horizontal velocity $v_x(t)$.

(d) Find the horizontal distance $x(t)$ that the projectile travels by time t.

(e) Express z in terms of x.

(f) Plot the trajectory $z(x)$ of the projectile for your choice of values for $h, v_x(0), v_z(0)$ and with $g \cong 10\ m\ s^{-2}$.

5. **Gelibolu / Gallipoli**: In the First World War / Dardanelles Campaign (Gelibolu), "new" technology guns were used. These guns fired bullets with initial speed $v_0 = 800\ m/s$. To hit a target at a distance D meters, a faster bullet is aimed at a lower angle. So the bullets crossed the battlefield on low trajectories and caused many casualties.

   Let us work this out:

   (a) What is the time for a bullet fired at a speed v_0 and angle θ from the horizontal to reach its maximum height, h?

   (b) What is the maximum height, h?

   (c) What is the range D of the bullet, in terms of v_0 and the angle θ?

   (d) Express the angle θ needed to reach a range D in terms of D, g and v_0. Assume θ is very small and use the approximation $sin(2\theta) = 2\theta$ (for θ in radians).

   (e) Express h in terms of D, g and v_0.

   (f) For $D = 100\ m$, $v_0 = 800\ m/s$ and $g = 10\ m/^2$, find h.

6. A small airplane is flying with a velocity of $100\ m/s$. A sudden wind blowing from behind gives it a constant acceleration of $a = 5t\ m/s^2$ for 5 seconds.

   (a) What is the new velocity of the plane?

   (b) How far does the plane go during this 5 second interval?

7. Ahmet's car accelerates for 10 seconds on a straight line, starting from rest, with a time dependent acceleration $a(t) = 2t/3\ m/s^2$.

   (a) What is the acceleration at $t = 3\ s,\ 6\ s,\ 9\ s$?

   (b) What is the velocity, in m/s, at $t = 3\ s,\ 6\ s,\ 9\ s$?

   (c) What is the distance the car has traveled, by $t = 3\ s,\ 6\ s,\ 9\ s$?

   (d) What is the average acceleration?

8. The number of cars entering the Bosphorus bridge to cross it, in each minute between 5 pm and 6 pm is given by the function

$$n(t) = 100 + 60\left(\frac{t}{60}\right)^2$$

   where t is the time in minutes since 5 pm. What is the total number of cars entering the bridge between 5 pm and 6 pm?

9. A car starts from rest at $t = 0$ and accelerates to a speed of 60 km/hr in 1 minute along a straight road. The acceleration is constant. The car then moves at constant speed, 60 km/hr, for 10 minutes. It then decelerates at the same constant rate as the acceleration, and stops after 1 minute of deceleration.

    (a) What is the magnitude of the acceleration and deceleration in meters per minute squared, m/min^2 ?

    (b) Make a graph of the velocity $v(t)$ in m/min versus time $t(min)$.

    (c) Calculate the total distance the car has traveled, in meters, geometrically, by calculating the area under this graph of $v(t)$.

    (d) Write the expressions, piecewise, for the function $v(t)$ between $t = 0$ and $t = 12$ min.

    (e) Calculate the total distance the car has traveled, in meters, by integrating the function $v(t)$.

10. If an airplane is initially moving in the x-direction, so $\mathbf{v} = 100\ m/s\ \mathbf{i}$, and the wind starts to blow, at time $t = 0$, with an angle such that it gives the plane an acceleration of $\mathbf{a} = (0.3t^2\mathbf{i} + 0.4t\ \mathbf{j})\ m/s^2$

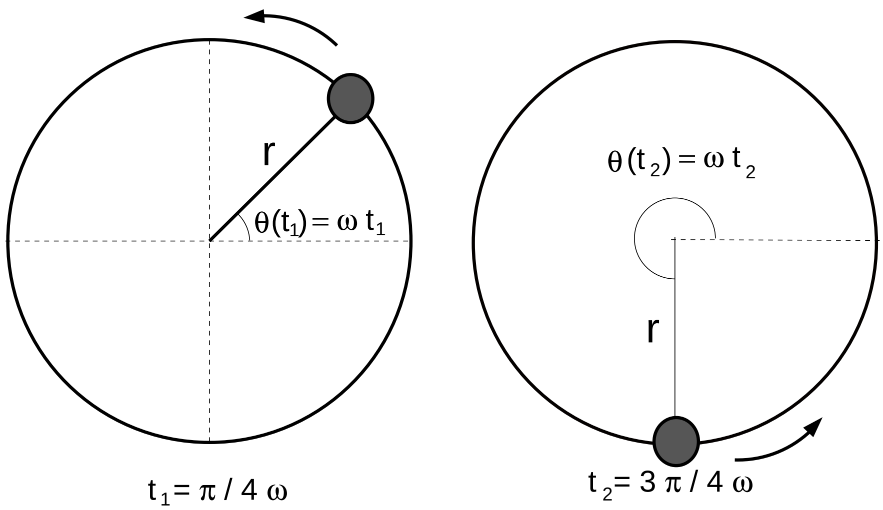
    (a) What is the velocity of the airplane at $t = 5\ s$?

    (b) What is the speed at $t = 5\ s$?

    (c) What is the magnitude of the acceleration at $t = 5\ s$?

Chapter 7

Circular motion



7.1 Uniform circular motion

This figure shows a particle moving on a circle[1]. It is moving with *constant speed* v. Circular motion at constant speed is called "*uniform circular motion*". In this example the particle is moving in the *counter clockwise* direction. Choose time $t = 0$ as a time when the particle is on the positive x axis. The "*angular position*" $\theta(t)$ measured in the counterclockwise direction from the initial position on the x axis gives the position of the particle on the circle at time t. We denote the distance the particle has moved along the circle, by time t, as the "*length of arc*" $s(t)$. What units shall we choose for the angle θ ? In the conventional unit of degrees, denoted ^{0}, a full revolution around the circle is 360^0 and a right angle is 90^0. The length of arc the particle travels on the circle increases in proportion to the angle θ. It is also proportional to the radius r of the circle: For the same angle θ, a particle traveling on a larger circle will have traversed a longer distance s. Thus, $s = constant r\theta$, where the constant of proportionality depends on the units we choose for θ. The simplest unit for θ is obtained by choosing the constant to be 1, so that $s = r\theta$. The full length of arc around the

[1]Check out the simulation in NS101-SUCourse for circular motion.

circle, the circumference is $s = 2\pi r$ - so in our new unit for θ, the *radian*, the full revolution is

$$\boldsymbol{2\pi} \textbf{ radians} = \textbf{360}^\circ, \quad \textbf{1 rad} = \frac{\textbf{180}^\circ}{\boldsymbol{\pi}}$$

As the particle moves with constant speed v along its circular path the angle $\theta(t)$ is also increasing at a constant rate ω, called the *angular velocity* or *radian velocity*,

$$\theta = \omega t \tag{7.1}$$

This is a velocity because it has a direction. The angular velocity ω is positive when the motion is in counter clockwise direction, the direction of increasing θ; for clockwise motion ω is negative. We consider circular motion in one given direction, say counter-clockwise, and will not distinguish between angular velocity and angular speed. The unit of ω is *radians/second*. The length of arc traveled in time t is

$$s(t) = vt = r\theta(t) = r\omega t, \tag{7.2}$$

relating the speed v to the angular speed ω

$$v(m/s) = \omega(rad/sec)r(m). \tag{7.3}$$

The particle comes back to its starting point on the x-axis after a time P, the "*period*". Circular motion, like any motion *on a closed orbit*, is periodic: it repeats itself, going back through the same point again and again, once every time period P, such that the speed is

$$v = \frac{2\pi r}{P} \tag{7.4}$$

The constant angular speed ω in uniform circular motion is

$$\omega = \frac{2\pi}{P} \tag{7.5}$$

in units of rad/s - the full revolution of 2π rad is covered in one period, of P s.

Solved Problem: Angular Velocity

One is driving on a circular path and reads a constant speed of 72 km/hr on the speedometer. The driver completes one rotation in 1 min.

1. What is the radius of the circle? Take $\pi = 3$.

   First, list what we know: $v = 72$ km/h, and $P = 1$ min $= 60$ s. We need to convert v to SI units:

$$v = 72\frac{\cancel{km}}{\cancel{h}}\,\frac{1000\ m}{1\ \cancel{km}}\,\frac{1\ \cancel{h}}{3600\ s} = 20\ m/s$$

   The radius r is related to v and P as $2\pi r = Pv$, so:

$$r = \frac{Pv}{2\pi} = \frac{(60\ s)(20\ m/s)}{2\pi} = 200\ m$$

2. What is the angular speed of the car?

$$\omega = \frac{r}{v} = \frac{200\ m}{20\ m/s} = 10\ rad/s$$

3. If the car is moving with the same speed but around a circle with a smaller radius, what can you say about the angular speed of the car?

   The car still moves 20 m every second on the smaller circle, but since the circumference of the circle is much shorter, 20 m takes up a larger fraction of the circumference. This also means that the car travels a larger angles every second, hence the angular speed (angles per second) is larger compared to the larger circle case.

7.2 Position, velocity and acceleration in uniform circular motion

The position of the particle is given by the position vector

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} = r\cos(\omega t)\mathbf{i} + r\sin(\omega t)\mathbf{j} \tag{7.6}$$

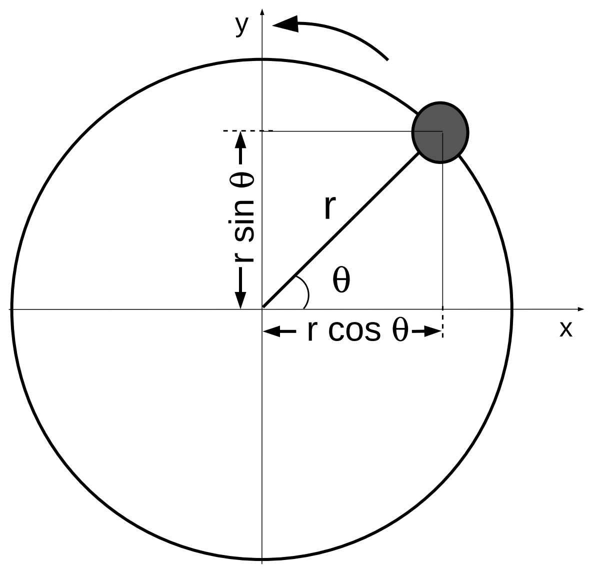
$\mathbf{i}$ and $\mathbf{j}$ are unit vectors in the x and y directions respectively.
The velocity vector is:

$$\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt} = \frac{dx(t)}{dt}\mathbf{i} + \frac{dy(t)}{dt}\mathbf{j} = r\left[\frac{d\cos(\omega t)}{dt}\mathbf{i} + \frac{d\sin(\omega t)}{dt}\mathbf{j}\right] \tag{7.7}$$

where we have used the fact that for motion on a circle, r is constant. In order to determine the velocity vector fully, one now needs to calculate $d\cos(\omega t)/dt$ and $d\sin(\omega t)/dt$.

Now the rate of change of the sine or cosine of the angle as time goes on is simply *the rate of change of the sine or cosine due to the change of the angle, times the rate of change of the angle with respect to time.* For *uniform* circular motion the angle θ increases at the constant rate

$$\frac{d\theta}{dt} = \omega \tag{7.8}$$



So we obtain

$$\frac{d\cos(\omega t)}{dt} = \left[\frac{d\cos(\omega t)}{d(\omega t)}\right]\cdot\left[\frac{d(\omega t)}{dt}\right] = \omega\left[\frac{d\cos(\omega t)}{d(\omega t)}\right] \tag{7.9}$$

$$\frac{d\sin(\omega t)}{dt} = \left[\frac{d\sin(\omega t)}{d(\omega t)}\right]\cdot\left[\frac{d(\omega t)}{dt}\right] = \omega\left[\frac{d\sin(\omega t)}{d(\omega t)}\right] \tag{7.10}$$

We need to understand how the cosine and sine of the angle $\theta = \omega t$ change as the angle changes. Using Figure 7.1, we find that

$$\frac{d\cos(\theta)}{d\theta} = -\sin(\theta) \tag{7.11}$$

$$\frac{d\sin(\theta)}{d\theta} = \cos(\theta) \tag{7.12}$$

So we have learned something important: how to take the derivatives of the sine and cosine functions! This is a piece of mathematics which we can use not only for circular motion but for other subjects also. Let us now continue with circular motion. Putting everything together, we get the velocity vector for uniform circular motion at all times:

$$\mathbf{v}(t) = \omega r\ [-\sin(\omega t)\mathbf{i} + \cos(\omega t)\mathbf{j}] \tag{7.13}$$

The speed is just ωr and the direction of the velocity is perpendicular to r. The velocity is at the direction of *the tangent* to the circle. You could have figured this out from the picture. Taking the time derivative of the position vector $\mathbf{r}(t)$ of course gives the same correct result.

Now that we know how to handle derivatives of the sine and the cosine, it is straightforward

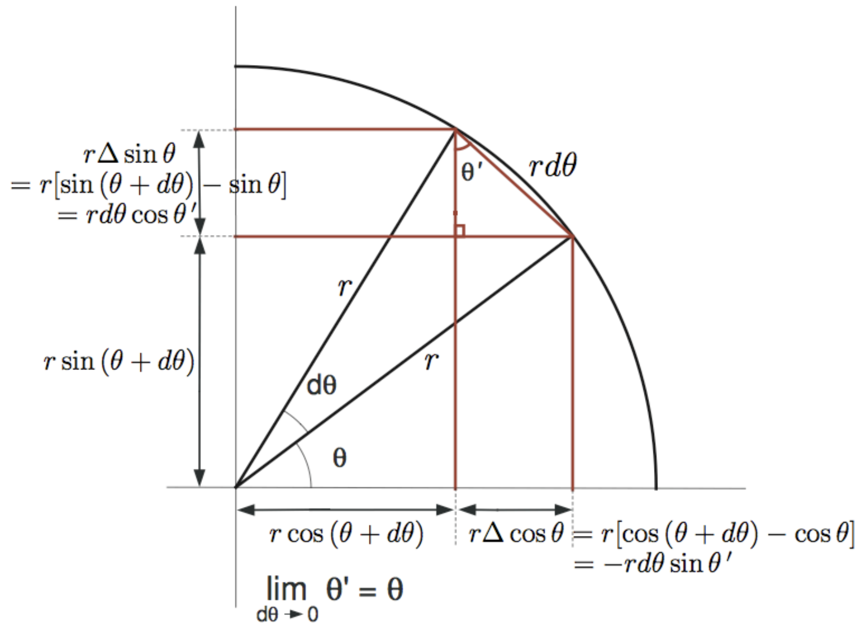


Figure 7.1:

to calculate the acceleration:

$$\mathbf{a}(t) = \frac{d\mathbf{v}(t)}{dt} = \omega r \left[\frac{-d\sin(\omega t)}{dt}\mathbf{i} + \frac{d\cos(\omega t)}{dt}\mathbf{j} \right]$$

$$\mathbf{a}(t) = -\omega^2 r\, [\cos(\omega t)\mathbf{i} + \sin(\omega t)\mathbf{j}] = -\omega^2 \mathbf{r}(t)$$

So motion on a circle at uniform angular speed ω (with speed $v = \omega r$) *requires an acceleration* directed in the $-\mathbf{r}(t)$ direction, that means *towards the center of the circle*; the magnitude of the acceleration is

$$a = \omega^2 r = v^2/r \tag{7.14}$$

According to Newton's Second Law an acceleration towards the center will take place only if *there is a force towards the center.* For "*heavenly bodies*" (stars, planets in circular orbits) this force is *gravitational attraction.* Motion in a circle is not just *the natural "perfect" state* as Aristotle and Ptolemy thought. The orbits of stars and planets are governed by the Universal Gravitational Force, as Newton discovered from Kepler's Laws for the orbits of the planets.

For the planets in our Solar System, the orbit is an ellipse rather than a circle. The acceleration is provided by gravitation. The calculations are similar to those for circular motion, but a bit more complicated because in an ellipse the distance r (the magnitude of the position vector) also changes in time.

CHAPTER 7 - PROBLEMS:

1. Evaluate the angles $45°, 60°, 90°, 180°, 270°$ in radians.

2. An audio CD rotates at about 300 rpm ("rotations per minute") on average. The diameter of a CD is 12 cm.

   (a) what is the period of rotation of the CD?

   (b) what is the angular speed of the CD in rad/sec?

   (c) what is the linear speed of a point on the rim of the CD?

3. A child on a merry-go-round (atlıkarınca) has the coordinates $x = r\cos\omega t$, $y = r\sin\omega t$, $z = z_0(1 + (1/2)\sin 3\omega t)$ where $\omega = 2\pi/T$ and $T = 6$ seconds.

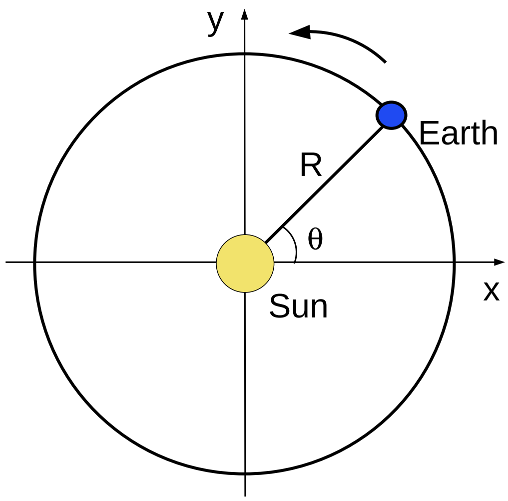
   (a) Write the position vector.

   (b) Write the velocity vector.

   (c) Write the acceleration vector.

   (d) What is the speed as a function of time?

   (e) Describe the 3 dimensional motion of the child on the merry-go-round.



4. The Earth revolves around the Sun almost in a circular orbit. The Sun is approximately 150 million kilometres away from the Earth.

   (a) What is the period of rotation?

   (b) What is the speed of the Earth orbiting the Sun?

   (c) What is the acceleration of the Earth orbiting the Sun?

   (d) If the Earth is aligned with the positive x axis at $t = 0$ and the direction of rotation is as seen in the figure. Write $r(t)$, $v(t)$ and $a(t)$.

   (e) Plot the x vs t graph for the motion of Earth.

5. Ahmet is driving at a speed of 36π km/h. He makes a sharp turn of 90 degrees in 10 seconds, following a circular path without reducing his speed.

   (a) What is the radius of the circle?

   (b) What is the magnitude of his acceleration? Compare this with the gravitational acceleration g.

   (c) What is the direction of his acceleration?

   (d) What is the magnitude and direction of his average acceleration during this motion?

Chapter 8

Newton's Laws of Motion

After the systematic experimental work of Galileo [1], motion was understood much better. The roles of mass (inertia), gravity, friction, velocity and acceleration became clear. Newton realized that the *acceleration*, rather than the velocity, of a given body is determined by the effects of other bodies (*forces*). Under a given force the acceleration is less for a more massive body, in inverse proportion to the mass. The acceleration at each moment determines the velocity and position at later times. In addition to developing a systematic understanding of physics, in order to formulate his Laws of Motion, Newton had to invent a new branch of mathematics, the Calculus, (along with, but independently from his contemporary, the German mathematician and philosopher Leibniz).

Newton's First Law, *"the Law of Inertia" or "the Law of Conservation of Momentum"* states that when no forces are acting on it, which means when it is completely isolated from all other bodies, an object moves with constant *momentum vector* **p**:

$$\mathbf{p} \equiv m\mathbf{v} = \text{constant}$$
$$\frac{d\mathbf{p}}{dt} = 0, \qquad \text{when} \quad \mathbf{F_{tot}} = \mathbf{0}. \tag{8.1}$$

Here m denotes the mass, $\mathbf{v}$ the velocity vector, $\mathbf{p}$ the momentum vector and $\mathbf{F_{tot}}$ denotes the total (or net) force on the body. Usually the mass of the object remains constant during the motion, so the First Law just states that when no net force is acting on it, an object moves with constant speed, on a straight line, in the same direction. *If the object breaks up into pieces or loses mass, as in the case of a jet airplane, the correct form of the law, as shown by experiments, is that the total momentum remains constant.* We will discuss conservation of momentum and the conditions under which it holds in more detail in the next chapter.

Question: Why is this law called *"the Law of Inertia"* ?

Newton's Second Law, *"the Equation of Motion"* describes how interactions with other bodies affect the motion of a body:

$$\begin{aligned} \mathbf{F_{tot}} &= \frac{d\mathbf{p}}{dt} \\ \mathbf{F_{tot}} &= m\frac{d\mathbf{v}}{dt} = m\mathbf{a}. \end{aligned} \tag{8.2}$$

[1] see Gribbin, Chapter 3

The last equation is obtained when the mass m is constant, as is usually the case. This is the famous $F = ma$ law. This law recognizes that interactions with other bodies (*"forces"*) affects motion by determining the acceleration, not the velocity. Before Galileo and Newton, forces were supposed to cause motion by determining velocity. This was partly due to misconceptions about friction forces. The development of the scientific method, and the careful and systematic experimental investigations of Galileo on friction and gravity led the way to Newton's Second Law. The First Law, which is a special case of the Second Law, refers to motion at constant velocity, *including the case of rest,* $v = 0$.

Most ancient and medieval philosophers and "Natural Philosophers" before Newton considered the state of rest and the state of motion at constant velocity to be different states. The state of rest was supposed to be the *natural state* of all bodies, while the state of motion, even at constant velocity, required forces. According to Newton's Laws, motion at constant velocity means the total force is zero. There may be no forces at all, or the forces balance each other to give a total force of zero. Thus constant velocity situations in many cases are due to a balance between friction and other forces.

The first good experimental measurements of velocity and acceleration, and of time, were made by Galileo. The *quantitative* description of instantaneous velocity became possible only with Newton's development of the Calculus. Before these developments the state of motion with constant velocity and with acceleration were not distinguished clearly. *Motion with constant velocity, which requires a balance between friction and other forces, was easily generalized to all motion, with the misconception that velocity requires force.*

For forces other than gravitation, careful experiments on bodies of different mass under the same force verify that the resulting acceleration is inversely proportional to the mass, $F = ma$. The Second Law actually *defines* force, as a precise measure of the effect on a given body due to other bodies with which it interacts. Acceleration is a precisely measurable quantity. So is mass, defined as the *quantity of matter.* Thus the Second Law gives the force unit in the SI system as the $Newton$ (N), which is the force that accelerates a mass of 1 kg by 1 m/s^2; 1 $N = 1\ kg\ m/s^2$.

Question: What does "the quantity of matter" measure? In practical terms: How is it possible to measure, actually to *count*, the quantity of matter? What are the natural units of mass?

Newton's Third Law states that the force applied by body 1 on body 2, $\mathbf{F_{1,2}}$, is equal in magnitude but opposite in direction to the force applied by body 2 on body 1, $\mathbf{F_{2,1}}$; *"Action equals reaction"*:

$$\mathbf{F_{1,2}} = -\mathbf{F_{2,1}}. \tag{8.3}$$

When two bodies are interacting, the motion of each body is determined by putting the force acting *on that body.* The two equations of motion are, according to Newton's Second Law,

$$\begin{aligned} \mathbf{F_{2,1}} + other\ forces\ on\ 1 &= m_1\mathbf{a_1} \\ \mathbf{F_{1,2}} + other\ forces\ on\ 2 &= m_2\mathbf{a_2}. \end{aligned} \tag{8.4}$$

8.1 Forces Acting on a Body

The *total* force acting on a body due to all other bodies determines the acceleration, according to Newton's Second Law. From the components of the acceleration vector, one can get, by integrating, the components of velocity, and by integrating once again, the displacement, as a function of time. Thus the *trajectory* is determined from the initial position and velocity of the body, and Newton's Second Law, with a knowledge of all the forces acting on the body at each instant. The forces acting on a body depend on its position and sometimes also on its velocity. In solving the equation of motion (Newton's Second Law) one must take into account the changing forces as the body moves along.

For an *extended*[2] body with many parts, this program is complicated because one needs to solve the equation of motion for each piece of the body. For rigid bodies it is enough to follow the motion of the *"center of mass"*, which we will define in the next chapter, together with the rotation of the body with respect to the center of mass. We will therefore treat the motions of extended rigid bodies as if these bodies were point particles, associating the mass of the body with the motion of its center of mass. The *mechanics of fluids* entails many new concepts and tools, to follow a range of fascinating kinds of behavior, like *turbulence*, which are not completely understood.

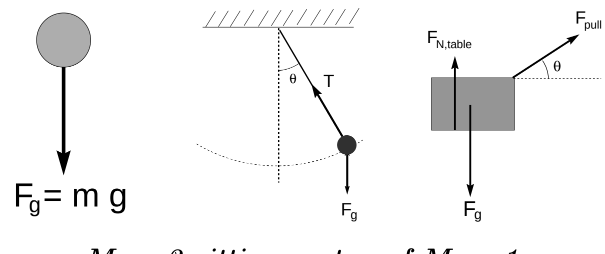
It is useful to relate the resultant acceleration of a body at a particular moment, or in a particular situation, to the resultant total force by showing, in a picture, the vectors for all the different forces acting *on that body.* This is called a "*free body diagram*". One can then take the components of all the forces on each coordinate axis and equate the total to $m\mathbf{a}$ along that axis.

Drawing Free-Body Diagrams

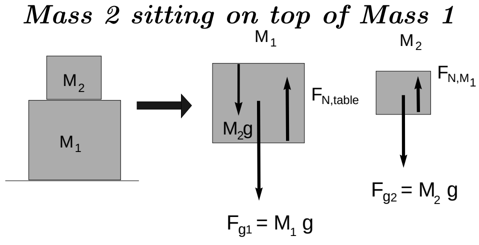
A free-body diagram shows *all* the forces acting *on* the body of interest. These external forces shown in the diagram determine the motion of the body. The forces exerted *by* the body must not be included, as they do not affect the motion of the body itself.

Examples of Free Body Diagrams:

Free-falling ball ***Pendulum*** ***Box being pulled***



Mass 2 sitting on top of Mass 1



[2]extended = not small enough to be considered as if it is a point

Fundamental forces like the *electric and magnetic forces* and the *gravitational force* are forces acting *at a distance*. This means the two bodies do not have to touch each other for the force to act. Combinations of electromagnetic forces from different objects, like, for example, the total force on a particular electron due to all the electrons and protons in neighboring atoms, are also forces that act at a distance. Among forces acting between macroscopic bodies there are some that are *contact forces*. These arise only when the two bodies are actually in contact. Among contact forces is the so called *normal* force, due to the rigidity of macroscopic bodies which cannot penetrate into each other. The *surface friction force* exerted on a body across its surface of contact with another body is also a contact force. When viewed microscopically, normal forces and friction forces are not really contact forces. They arise as the effective macroscopic total of electromagnetic forces of the atoms of either body across the surface of contact. *Like all forces that determine the properties of materials, these forces microscopically arise from a complex superposition of electromagnetic forces from many atoms.* Macroscopic contact forces are microscopically not contact forces at all. In an average sense atoms in a solid or liquid are closely packed, so one may say they are in contact, but at the atomic, quantum mechanical level, "contact" has an altogether different nature. Furthermore electrons are distributed over the atom, over a volume that is 10^{15} times that of the nucleus that contains the protons. Thus there is no contact between the individual electrons and protons that jointly make up the total electromagnetically generated microscopic force.

Unlike the fundamental forces acting at a distance, neither the normal force nor the friction force have generally valid formulas. Rather, these forces are very dependent on the particular situation, on the details of the surface structure on both bodies, and in the case of friction, also on the relative velocity of the two bodies. Friction forces are directed oppositely to the direction of motion of the body they act on. The magnitude of the friction force can be constant, as is the case, to a good approximation, for a block sliding over an inclined plane. In other cases, the magnitude of the friction or damping forces may be proportional to the relative speed, or to its square, or some other function of velocity. Constant friction forces across a surface, the static and kinetic friction forces, are just proportional in magnitude to the normal force in many common situations. The normal force in turn is supplied by the rigidity of the materials in response to other forces present, such that the resultant component of acceleration perpendicular (in other words, *normal*) to the surface is zero. Thus, through the microscopic response and adjustment of the atoms near the contact surface, the normal force ensures that the bodies do not start to move into each other. The magnitude of the normal force in a given situation is inferred from all the other forces perpendicular to the surface and the net acceleration perpendicular to the surface, zero or nonzero, in the given situation. In motion on a circular track, the magnitude of the normal force is just right so as to supply, together with all the other forces, the component of the acceleration towards the center of the circle, $v^2\,/r$. The normal force is supplied by rigidity up to a very high *yield threshold*. Below the threshold the normal force increases in response to external forces, to just the right strength to give zero total acceleration across the surface, into the rigid body. If the other forces on the surface are stronger than the threshold, the surface will yield, the material may break, collapse, or even liquefy.

8.2 What is Weight?

Weight is the force that the Earth applies on a body on its surface. As we will learn in Chapter 14, one of the fundamental interactions in nature is the universal gravitational force between any two masses. The force that the Earth, of mass M_E, exerts on an object of mass m on its surface is called the *weight* of that object

$$\mathbf{W} = -\frac{GM_E m\hat{\mathbf{R}}}{R^2} \tag{8.5}$$

where R is the distance of the object from the center of the Earth, which is just the radius of the Earth. Here G is a constant of proportionality called Newton's universal constant of gravitation. The value of G in SI units is $6.67 \times 10^{-11} m^3 kg^{-1} s^{-2}$. We denote by $\mathbf{R}$ the position vector of the point mass m from the center of the Earth. This force that the Earth exerts on the mass m is an attractive force, directed towards the center of the Earth, as the direction $\hat{\mathbf{R}}$ indicates. For an object on the surface of the Earth R is just the radius of the Earth, $R \cong 6371km$ – this is the average radius of the Earth assuming it is a sphere. The weight is proportional to the mass m

$$W = \frac{GM_E m}{R^2} \equiv mg$$

defining the gravitational acceleration $\mathbf{g}$ on the surface of the Earth, with magnitude

$$g \equiv \frac{GM_E}{R^2} \cong 9.8ms^{-2}, \tag{8.6}$$

directed towards the center of the Earth. The gravitational acceleration on the surface of the Earth is the same for all bodies, as it does not depend on the mass m of the body.

8.3 Normal Forces

Figure 8.1 shows the normal forces (F_N) in three situations.

In the upper panel, the free body diagram for the block shows the force that the Earth applies on the block, which is just the block's weight (mg), and the normal force that the material under the horizontal surface (say a table in your kitchen) applies on the block. Since the block remains at rest, its vertical acceleration is zero. This means the downward force applied by the Earth (the weight of the block) and the upward normal force applied by the table are equal in magnitude and cancel exactly. So far we have applied Newton's 2nd Law for the block.

Incidentally, if we were interested in the equation of motion for the *table* we would use the force applied *by* the block *on* the table in Newton's 2nd Law. By Newton's 3rd law, the normal force that the block applies on the material object under the surface is equal in magnitude and opposite in direction to the normal force that the kitchen table applies on the block. Thus the normal force applied by the block on the kitchen table, which is due to the material structure of both the block and the table, *happens to be* equal to the weight of the block. Both of the normal forces applied by block on table, and by table on block are material forces. They are not gravitational forces. Their magnitudes happen to coincide with the block's weight in the present case only because the material forces arise in response

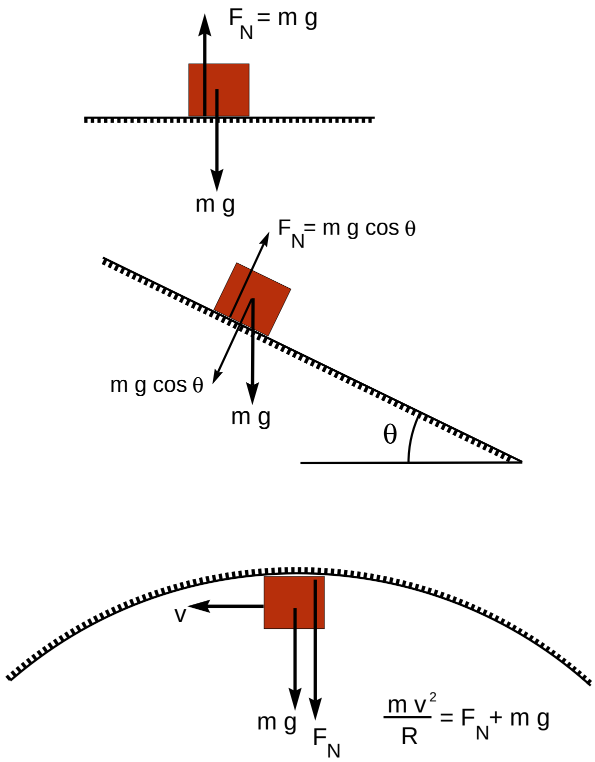


Figure 8.1:

to gravitational forces.

For our second example, shown in the middle panel in Figure 8.1, the block is on a frictionless *inclined plane*. The block can move and be accelerated *along* the inclined plane. But the components of its velocity and acceleration *perpendicular to* the inclined plane, *into or out of* the inclined plane, are zero as long as the plane and block remain rigid materially. This means that the total force in the normal direction is zero. The normal force that the material of the inclined plane applies on the block across the surface of contact must cancel exactly, so it must have the same magnitude as, the *component* of the force that the Earth applies on the bock, the block's weight, that is *perpendicular* to the surface, which is $mg \sin\theta$. This is the magnitude of the normal force.

In the third example, shown in the bottom panel in Figure 8.1, the block is moving on a frictionless vertical circular track with speed v.

How does the block stay on the track?

Let us consider all the forces on the block. There is the normal force applied on the block by the material of the circular track. This force keeps the block from moving into the material of the track. At the moment shown this normal force is downward. In addition there is the force that the Earth applies on the block, which is just the weight of the block. Both forces are downward at the instant seen in the figure. They add up in Newton's 2nd law to give a net downward acceleration

$$F_{total} = F_N + mg = m\ a_R \tag{8.7}$$

where a_R is the acceleration in the downward vertical direction, which, at this moment in the motion along the circular track, happens to be the radial direction towards the center. For uniform circular motion we have learned in Chapter 7, the acceleration, in the radial direction towards the center of the circle, is just $a_R = v^2/R$, where R is the radius of the track. So the magnitude of the normal force supplied by the material of the circular track must satisfy

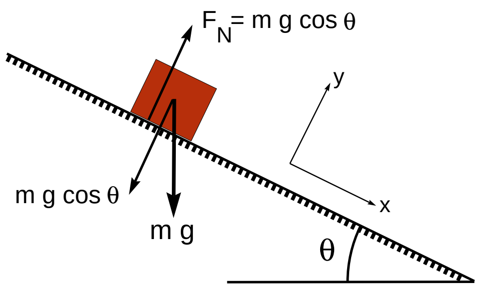
$$F_N + mg = \frac{mv^2}{R}. \tag{8.8}$$

This gives the normal force for this particular situation. There is no general formula for the normal force; for each problem, writing Newton's Second Law for the direction perpendicular to the surface, including all other forces in that problem, along with the normal force, and noting the value of the acceleration in the normal direction, one deduces the normal force for that problem.

Sometimes you hear of the *so-called "centrifugal" or "centripetal" forces.* This is misleading. What is meant is only the right hand side, the ma part of $F = ma$, Newton"s Second Law, as in Equation 8.7, for the special case of uniform circular motion. There is no force of nature that is responsible specifically and only for circular motion. It is the total of some combination of real forces, like gravity, electrostatic attraction, friction, tension in a rope, normal force applied by a rigid surface etc, whichever are present in a particular situation, appearing in the F part of $F = ma$, that add up to $ma = mv^2\,/r$ to sustain uniform circular motion.

Equation of Motion (Newton's 2nd Law) in 2 Dimensions

If the problem involves two dimensional motion, the equation of motion must be written for each dimension. For example, in the case of the block on a frictionless inclined plane:



We first define the x-y coordinates – here, we take $+x$ to be down the plane, and $+y$ to be upward perpendicular to the plane. You can choose your coordinates whichever way you like, as long as you stick to the same definition of coordinates throughout your calculation. It is a good idea to choose the coordinates in a way to make the statement and solution of the problem easy. In this example, taking y vertical and x horizontal is also OK, but the equations will be slightly more complicated. We choose x along the inclined plane because we know this

Continues on next page...

...Continued from previous page

is the direction of the motion. Then, the equations of motion in x and y are:

$$\begin{aligned} \Sigma\mathbf{F_x} &= m\mathbf{a}_x \\ \Sigma\mathbf{F_y} &= m\mathbf{a}_y \end{aligned}$$

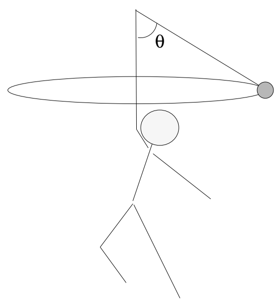
Using the figure, we find that their magnitudes are:

$$\begin{aligned} \Sigma F_x &= mg\sin\theta = ma_x \\ \Sigma F_y &= F_N - mg\cos\theta = ma_y = 0 \end{aligned}$$

The block would not move along the y direction, so we can immediately say that $a_y = 0$. Now, from these two equations, we find that $a_x = g\sin\theta$ and $F_N = mg\cos\theta$.

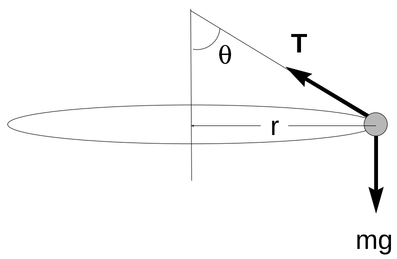
Solved Problem: Rotating Ball

A child is rotating a ball tied to a string over her head, as shown in the figure. She realizes that the faster she rotates ball goes higher and the angle θ gets larger. Length of the string is L.



1. Draw the free body diagram for the ball.

   There are only two physical forces acting on the ball, exerted by gravity (mg downward) and by the string (tension, T, along the string). The free-body diagram shows these two forces.

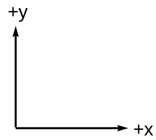


Continues on next page...

...Continued from previous page

2. Write the equation of motion.

   Since this involves 2-dimensional motion (x and y), we need to first define the x-y coordinates, and write the equations of motion in 2D. If we take the coordinates as shown, then the equations of motion in x and y are:



$$\Sigma \mathrm{F_x} = -T\sin\theta = ma_{\mathrm{circ}} = m\left(-\frac{v^2}{r}\right)$$
$$\Sigma \mathrm{F_y} = T\cos\theta - mg = ma_y = 0,$$

   where v is the speed of the ball in a circular motion, and r is the radius of the curvature of ball's rotation. r can be found in terms of L,
   $r = L\sin\theta$.
   The minus sign ($a_{\mathrm{circ}} = -\frac{v^2}{r}$) is there because the acceleration is towards the center of the circle, which in this case is in $-$x direction.

3. Find the speed of the ball where $\theta = 45°$.

   We can find the speed v as a function of θ by solving the two equations of motion above. From $\Sigma\mathrm{F_y}$ equation, we find $T = \frac{mg}{\cos\theta}$. Plug this into $\Sigma\mathrm{F_x}$ equation, we find:

$$-\cancel{m}g\frac{\sin\theta}{\cos\theta} = -\cancel{m}\frac{v^2}{L\sin\theta} \rightarrow v = \sqrt{gL\ \tan\theta\ \sin\theta}$$

   You can see that the speed of the ball depends on the radius as well as the angle. When $\theta = 45°$, $\tan 45° = 1$, $\sin 45° = 1/\sqrt{2}$, so $v(\theta = 45°) = \sqrt{gL/\sqrt{2}}$.

8.4 Friction

A *damping* (or frictional) force $\mathbf{F}_f$ on a body can depend on the shape of the body, on the density of air or other fluid around, on the roughness of the surface of the body and surfaces of any other rigid bodies with which it is in contact. A damping force usually also depends on the velocity $\mathbf{v}$; $\mathbf{F}_f = \mathbf{F}_f(\mathbf{v})$. The damping force $\mathbf{F}_f$ is a function of $\mathbf{v}$. Both its magnitude and its direction can depend on the magnitude and direction of $\mathbf{v}$. The direction of a damping force is opposite to the direction of motion.

As the simple and common example of a damping force we consider the friction force on a body sliding over a surface. In Figure 8.2 you see a block moving to the right on the horizontal surface. There is a force of friction acting on the block in the direction opposite

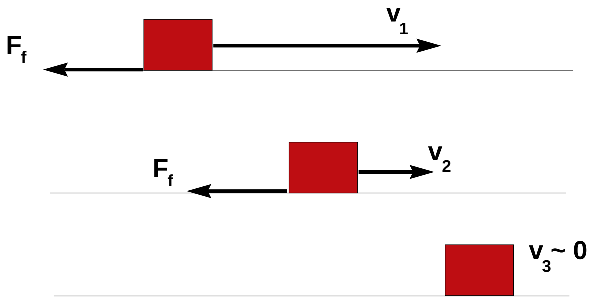


Figure 8.2:

to the direction of motion. The speed of the block decreases in time because of the damping force.

In many situations like this one, the friction force on a body sliding over a surface depends only on the direction of the velocity. Typically the constant magnitude of the friction force is proportional to the normal force exerted by the surface on the body, such that

$$\mathbf{F}_f = -\mu F_N \hat{\mathbf{v}}. \tag{8.9}$$

Here the coefficient of friction μ is a property of the surface, F_N is the normal force and $\hat{\mathbf{v}}$ is the unit vector in the direction of motion.

The coefficient of friction μ may depend on the relative speed between the bodies in contact. In many cases the coefficient of friction between two solid bodies is characterized by a constant value μ_k (coefficient of kinetic friction) valid to a good approximation for a large range of relative speeds starting from the lowest speed. The coefficient of friction has a higher value μ_s (coefficient of static friction) when the bodies in contact are not moving with respect to each other. So there are two different values of the coefficient of friction, with $\mu_s > \mu_k$ for most surface contacts. You have to apply a larger force ($\mu_s F_N$) to start an object at rest on a surface into motion. As soon as the object starts to move the magnitude of the friction force drops to $\mu_k F_N$.

CHAPTER 8 - PROBLEMS:

Conceptual Questions

Q1: A brick of mass m is sitting on a table. If you press down on the brick with your hand, will the normal force of the table greater than, smaller than, or equal to mg?

Q2: In general, the larger the normal force is, the larger is the friction force between the surface and the moving body. Why?

Q3: Does the friction force depend on the velocity vector of the body moving on the inclined plane?

1. Consider a block of mass m moving with uniform speed v on the frictionless vertical circular track as shown in the lower panel of Figure 8.1. What is the minimum speed needed for the block to stay on the track while going through the top point of the track if the radius of the track is R? Hint: The normal force exerted by the track is directed outwards from the surface of the track; it never points into the material of the track.

2. The block in Figure 8.1(lowest panel) is moving through a point on the track at an angular position α from the vertical, as measured from the center of the circle.

   (a) What is the normal force needed to keep the block moving through this point on the track at speed v ?

   (b) What is the minimum speed needed for the block to stay on the circular track while moving through this point?

3. Galileo showed that in the absence of friction all bodies fall with the same acceleration. A body falling freely from rest in vacuum falls a distance y in time t, given by the formula: $y = gt^2/2$. To show that this is true for all bodies, one needs to show that all bodies fall through the same distance y in the same time interval. To measure time intervals accurately was difficult in Galileo's time.

   (a) How do you measure time intervals in Experiment 1 of this course?

   (b) Take the distance an object falls freely in vacuum in 1 sec to be your unit of length. How many units of length does a body fall through in 1, 2, 3, 4, 5, 6 seconds?

4. Galileo used inclined planes to make time measurements easier: A frictionless inclined plane like the one shown in the middle panel of Figure 8.1 serves to slow down free fall.

   (a) Show all forces on a body of mass m sliding ("falling") down the inclined plane.

   (b) What is the total force along the direction perpendicular to the surface?

   (c) What is the magnitude of the "normal force" applied by the inclined plane on the body?

5. To "fall freely" for a distance s along a frictionless inclined plane takes a longer time t_{inc} compared to the free fall time t to fall the same distance s vertically in vacuum. For very small inclination angle α(in radians) of the plane,

(a) What is the value of t_{inc}/t if the angle α of the inclined plane from the horizontal is $\alpha = \pi/4\ radians = 45°$?

(b) Find an approximate expression for the ratio t_{inc}/t for small α, given that $\sin(\alpha) \cong \alpha$ for small angles α (in radians).

6. The normal force, weight and acceleration (D. Üçer):

   Weighing scales give your weight in kilograms. The kilogram is actually a unit of mass. Scales measure the force that you exert on them, which is equal to the *normal force* that the scale exerts on you. When you stand on a scale in a stationary room, the force that you exert on the scale is equal to your weight, $W_g = M\ g$, where M is your mass and g is the gravitational acceleration on the surface of the Earth.

   You would like to weigh yourself in an elevator. Suppose the scale shows W_g when the elevator is at rest. Try to answer the following questions without any calculations – use your intuition.

   Would you weigh equal to, less than or more than W_g in the following situations:

   (a) when the elevator is moving upward with a constant speed?

   (b) when the elevator is moving downward with a constant speed?

   (c) When the elevator is accelerating while moving downward?

   (d) When the elevator is decelerating while moving downward?

   (e) When the elevator is accelerating while moving upward?

   (f) When the elevator is decelerating while moving upward?

   Now let us think in terms of Newton's laws.

   (g) Draw the free body diagram for your body, show all the forces acting *on* you.

   (h) Write Newton's Second Law.

   (i) Now answer the questions from a) to f) using the free body diagram and Newton's Second Law in each case. Were your first answers right?

   (j) Check your result in the case when the elevator cable is broken and the elevator is falling freely. What do you expect to read on the scale then? Are your equations consistent with this answer?

7. (a) What is the horizontal force F needed, in combination with the tension T in the string and the weight of the pendulum, to keep the pendulum in Figure 8.3 in balance, at rest, in the position shown at angle θ from the vertical?

   (b) Can you maintain the same angle if you do not apply the horizontal force F but instead set the ball into rotation around the vertical axis ?

   (c) If you think that this would work, what would be the period of rotation?

8. If two blocks with different masses are released from the top of two identical frictionless inclined planes, which one would reach the bottom first?

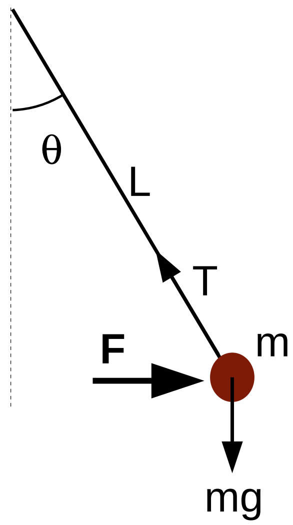


Figure 8.3:

9. Consider an inclined plane with inclination angle θ. The magnitude of the friction force is μ times the magnitude of the normal force, $F_f = \mu F_N$. Find the component of the total force along the plane on a block of mass M sliding down the inclined plane, and the acceleration down the plane. Note that the acceleration is the same for all bodies: it is independent of the mass M of the body.

10. A body is released from rest from the top of the inclined plane in Problem 9.

    (a) Find its speed t seconds after release.

    (b) Find the distance it has traveled along the incline by the time t.

    (c) When does the body reach the bottom of the incline if the height of the inclined plane is h?

11. A body is released from rest at the top of an inclined plane of height $1m$ and coefficient of friction $\mu = 0.1$. Find the time taken to reach the bottom if the angle of the incline is: a)30 degrees, b)45 degrees, c)60 degrees.

12. If two blocks with different masses $M_1 > M_2$ are released from the top of two identical inclined planes with friction, which one would reach the bottom first? The magnitude of the friction force on each of the blocks is proportional to the magnitude of the normal force that the inclined plane exerts on that block, $F_f = \mu F_N$.

13. For some surfaces, friction force F_f could be proportional to $(F_N)^2$ or $(F_N)^3$. Is the acceleration down an inclined plane with such surfaces still independent of the mass of the body?

14. Two identical blocks, Block 1 and Block 2 are released simultaneously from two frictionless inclined planes of inclination angles $\theta_1 = 45°$ and $\theta_2 = 30°$, and of the same height $h = 1\ m$. Which one will reach the end of the track first?

15. If both of the inclined planes in the previous problem had friction with the same coefficient of kinetic friction, $\mu = 0.1$, which of the blocks would reach the end of the track first?

16. You can call the acceleration you found in Problem 9 the effective gravitational acceleration g_{eff}.

    Make a table of g_{eff}/g for $\mu = 0$ and for $\mu = 0.1$, in each case for inclined planes of angle $\theta = 15°, 30°, 45°, 60°$.

17. Two masses (M_1 and M_2) are connected with a massless string. M_1 is on a table with a coefficient of friction μ, while M_2 hangs from a pulley without touching the surface (see Figure 8.4).

    (a) Draw a free-body diagram for each mass.

    (b) Find the acceleration of the masses.

    (c) Find the tension, T, on the string.

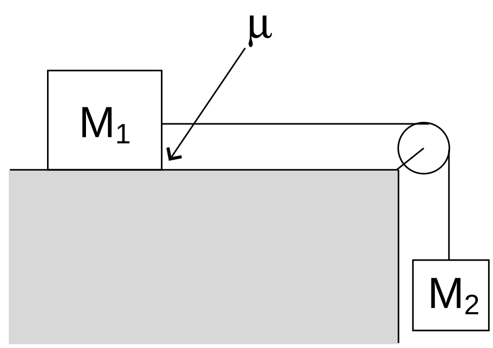


Figure 8.4:

Chapter 9

Momentum

Newton's Second Law leads to conservation laws under certain conditions. The Second Law states how an object will move under the effects of other bodies (forces). With the Second Law, if, in addition to all the forces acting on the body, the position and velocity of the body are known at a particular time t_0, its motion at all later times are determined. The total force gives the acceleration at each moment. The acceleration determines how the velocity changes. The new velocity determines where the particle will be at the "next instant".

The forces may depend on the position and maybe also on the velocity of the particle. At the new point where the particle arrives (with its new velocity), new values of the forces determine its acceleration, and the motion continues according to

$$\mathbf{F} = m\mathbf{a} = m\frac{d\mathbf{v}}{dt}. \tag{9.1}$$

Sometimes the mass of the object may change during the motion: The object may evaporate, explode, break into pieces, or, like a jet airplane or a rocket, may eject some mass. In such situations experiments show that the total force actually determines the rate of change of $m\mathbf{v}$ rather than of just the velocity $\mathbf{v}$:

$$\mathbf{F} = \frac{d(m\mathbf{v})}{dt} = \frac{d\mathbf{p}}{dt} = \frac{dm}{dt}\,\mathbf{v} + m\,\frac{d\mathbf{v}}{dt} \tag{9.2}$$

The quantity $\mathbf{p} = m\mathbf{v}$, called the "*momentum*" is the basic property whose rate of change is determined by the forces. Equation 9.2 is the correct general form of Newton's Second Law. Usually the mass is constant, so that the first term in Equation 9.2 drops and the familiar form of Newton's Second Law, Equation 9.1 is obtained.

Example: The force exerted by a truck moving at a speed of 60 km/h crashing into a car is much bigger than the force exerted by another car crashing into this car at 60 km/h. Mass matters as well as speed.

Question: Estimate the force at the impact when you crash into a glass door running at 6 km/h. The glass door stops you in about 2 sec (brings your speed from 6 km/h to 0 km/h in 2 sec).

When the total force on the object is zero it follows that "the momentum is conserved",

$$\frac{d\mathbf{p}}{dt} = 0. \tag{9.3}$$

So the momentum at any time t is the same as it was at the initial time t_0, when the motion was first observed:

$$\mathbf{p}(t) = \mathbf{p}(t_0) = constant$$

Let us now examine Newton's 2nd Law for a system made of many parts. It will be enough to consider just two bodies interacting with each other and also with the rest of the world at first: once we see how it works for a system with two components it is easy to generalize to three, four,...many body systems.

$$\frac{d\mathbf{p_1}}{dt} = \frac{d(m_1\mathbf{v_1})}{dt} = \mathbf{F}_{2,1} + \mathbf{F}_{ext,1} \tag{9.4}$$

$$\frac{d\mathbf{p_2}}{dt} = \frac{d(m_2\mathbf{v_2})}{dt} = \mathbf{F}_{1,2} + \mathbf{F}_{ext,2} \tag{9.5}$$

Here m_i is the mass, $\mathbf{v_i}$ is the velocity and $\mathbf{p_i}$ is the momentum of the body i (there are two bodies, so $i = 1, 2$).

$\mathbf{F}_{2,1}$ is the force that body 2 exerts on body 1, and $\mathbf{F}_{ext,1}$ is the total force that all objects other than body 2 (all external objects outside the system made of bodies 1 and 2) exert on body 1.

Similarly, $\mathbf{F}_{1,2}$ and $\mathbf{F}_{ext,2}$ are the forces on body 2 exerted by body 1, and by the rest of the world, respectively.

Newton's 3rd Law states that

$$\mathbf{F}_{1,2} = -\mathbf{F}_{2,1} \tag{9.6}$$

Using this and adding Equations (9.4) and (9.5), we find Newton's 2nd Law for the composite system made of bodies 1 and 2:

$$\frac{d\mathbf{p_1}}{dt} + \frac{d\mathbf{p_2}}{dt} = \mathbf{F}_{ext,1} + \mathbf{F}_{ext,2} \tag{9.7}$$

$$\frac{d\mathbf{p_{total}}}{dt} = \mathbf{F}_{ext,total} \tag{9.8}$$

Thus the total momentum $\mathbf{p}_{total} = \mathbf{p_1} + \mathbf{p_2}$ changes according to the total external force

$$\mathbf{F}_{ext,total} = \mathbf{F}_{ext,1} + \mathbf{F}_{ext,2}. \tag{9.9}$$

Internal forces between the parts of a system do not effect the total momentum; $\mathbf{p}_{total}$ does not change with time though $\mathbf{p_1}$ and $\mathbf{p_2}$ can both be changing. ***When the total external force is zero, then the total momentum of the system is conserved.***

9.1 Collisions

Consider an isolated system of two particles. *Isolated* means the total external force on this system is zero. So the total momentum is constant:

$$\mathbf{p}_{1,i} + \mathbf{p}_{2,i} = \mathbf{p}_{1,f} + \mathbf{p}_{2,f} \tag{9.10}$$

where i denotes the initial and f the final momenta of the two particles, 1 and 2. Let us take the initial situation with the two particles very far from each other but getting closer. Initially, when the two particles are very far from each other, the internal forces they apply on each other are very weak, approximately zero, $\mathbf{F}_{1,2} = -\mathbf{F}_{2,1} = 0$. This is true for almost all forces: when the particles are far from each other, the force is asymptotically zero. When the internal force on each particle is zero, the momentum of each particle is constant. So, when far away, both particles continue to move with constant momentum. As they get nearer, the internal forces they apply on each other become stronger. So both momenta start to change, in opposite directions, as the magnitude of the internal forces become stronger. As the particles get closer the forces get stronger and the momenta change at a faster rate. Now both particles are following curved (accelerated) paths. They do not fall into each other, even if the forces are attractive, because they have some sideways motion – as we will learn when we discuss the conservation of angular momentum. Eventually the curved path will lead the particles to move away from each other. As they get further the internal forces (the interaction) gets weaker. The momenta change less, and asymptotically become constant again when the particles are very far from each other, emerging from this encounter with the final momenta $\mathbf{p}_{1,f}$ and $\mathbf{p}_{2,f}$. Such an encounter is a *collision.*

Both momenta have changed during the collision:

$$\begin{aligned} \mathbf{\Delta p}_1 &= \mathbf{p}_{1,f} - \mathbf{p}_{1,i} \\ \mathbf{\Delta p}_2 &= \mathbf{p}_{2,f} - \mathbf{p}_{2,i}. \end{aligned}$$

From the conservation of momentum, it follows that

$$\mathbf{\Delta p}_2 = -\mathbf{\Delta p}_1 \tag{9.11}$$

The net change of momentum of particle 1, $\mathbf{\Delta p}_1$ which is due to the force applied on particle 1 by particle 2 is called *the impulse of particle 2 on particle 1.* Similarly, $\mathbf{\Delta p}_2$ is the impulse of particle 2 on particle 1. From Newton's Second Law, impulse can be calculated as:

$$\mathbf{\Delta p}_1 = \int_{t_i}^{t_f} \left(\frac{d\mathbf{p}_1}{dt}\right) dt = \int_{t_i}^{t_f} \mathbf{F}_{2,1} dt \tag{9.12}$$

From Newton's Third Law one can check that the impulse of particle 1 on particle 2 is just the negative of the impulse of particle 2 on particle 1:

$$\mathbf{\Delta p}_2 = \int_{t_i}^{t_f} \left(\frac{d\mathbf{p}_2}{dt}\right) dt = \int_{t_i}^{t_f} \mathbf{F}_{1,2} dt = \int_{t_i}^{t_f} -\mathbf{F}_{2,1} dt = -\mathbf{\Delta p}_1. \tag{9.13}$$

This can also be obtained directly from the conservation of total momentum (Equation 9.10).

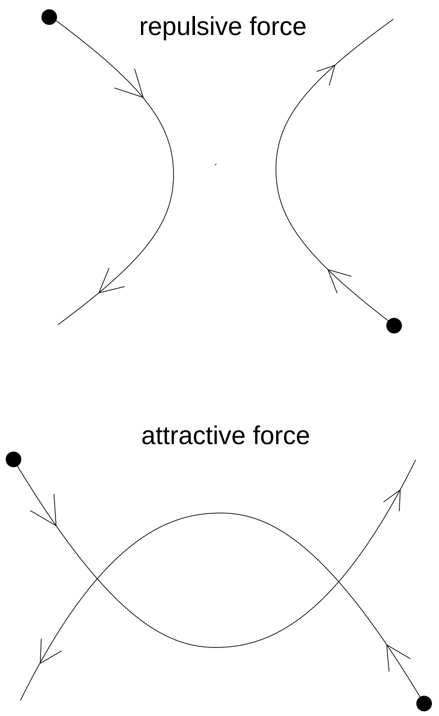


Figure 9.1:

9.2 * Center of Mass

The total momentum of the system is associated with the motion of the total mass. Thus one can define a velocity associated with the total momentum:

$$\mathbf{p}_{total} = (m_1\mathbf{v_1} + m_2\mathbf{v_2}) \equiv (m_1 + m_2)\mathbf{V}_{cm} \tag{9.14}$$

which gives

$$\mathbf{V}_{cm} = \frac{(m_1\mathbf{v_1} + m_2\mathbf{v_2})}{(m_1 + m_2)}. \tag{9.15}$$

This is the velocity of the "*center of mass*". The position of the center of mass $\mathbf{R}_{cm}$ is defined through

$$\mathbf{V}_{cm} = \frac{d}{dt}\left(\frac{m_1\mathbf{r_1} + m_2\mathbf{r_2}}{m_1 + m_2}\right) \equiv \frac{d\mathbf{R}_{cm}}{dt} \tag{9.16}$$

$$\mathbf{R}_{cm} \equiv \frac{m_1\mathbf{r_1} + m_2\mathbf{r_2}}{m_1 + m_2} \tag{9.17}$$

where $\mathbf{r_1}$ and $\mathbf{r_2}$ are the positions of the two bodies.

For a composite system the external forces determine the motion of the center of mass. The internal forces determine how the parts move with respect to the center of mass.

As an example, consider the Earth-Moon system. Both are attracted by the Sun's gravitational pull: There are external forces, applied by the Sun on both the Earth and the

Moon. The Earth-Moon system moves around the Sun under the attractive total external force. The center of mass of the Earth-Moon system is very close to the center of the Earth because the Earth is much more massive than the Moon. This center of mass moves in orbit around the Sun. The Moon and the Earth each move around the center of mass. The orbits are almost circular. Their motions around the center of mass are determined by the *internal* force of gravitational attraction *between* the Earth and the Moon. The Earth traces a tiny orbit, because its own center is so very close to the center of mass of the Earth-Moon system.

CHAPTER 9 - PROBLEMS:

1. A block of mass 1 kg (block 1) moving on a frictionless plane at initial velocity $\mathbf{v}_{1,i} = 20\ m/s\ \mathbf{i}$ hits another block (block 2), of the same mass, initially at rest. After the collision block 1 is moving with some final speed $\mathbf{v}_{1,f}$ at an angle of 45 degrees from its initial direction, so that $\mathbf{v}_{1,f} = v_{1,f}\sqrt{2}/2\ (\mathbf{i}+\mathbf{j})$ (see Figure 9.2). The collision is elastic, so the total kinetic energy is conserved. What are the final velocities (final speeds and the direction of motion of block 2)?

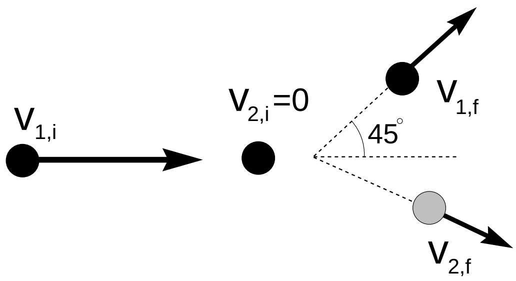


Figure 9.2:

2. A small rocket of mass 1 ton is moving horizontally with a velocity of 600 m/s in the x direction. The total external force on the rocket is zero. 10 kg of jet gas is ejected at a velocity (in the x direction) of -6000 m/s with respect to the ground (see Figure 9.3). The ejection takes place at a constant rate over a time interval $\Delta t = 1s$.

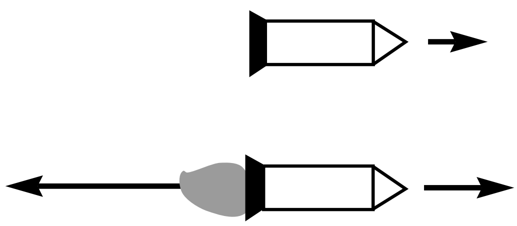


Figure 9.3:

(a) What is the velocity of the rocket after the jet is ejected ?

(b) What is the force (vector) that the jet exerts on the rocket during the ejection?

3. Generalizing from two bodies to many bodies:

   (a) Consider a system with three interacting bodies. Write the equations of motion for each body, including the internal forces between every pair of bodies and the external force on each. Repeat the arguments from Equations 9.4 to 9.9 for the three body system. Find the expression for its center of mass $\mathbf{R}_{cm}$.

   (b) Do the same for a system of N bodies.

4. Determine the location of the center of mass of the Earth-Moon system. How far is the center of mass from the center of mass of the Earth? From the center of mass of the Moon?

   The mass of the Earth is 6.0×10^{24} kilograms, the mass of the Moon is 7.4×10^{22} kilograms, the distance of the center of the Moon to the Earth's center is 384,400 km. Draw a figure to guide you in solving this problem. Take the origin to be at the center of the Earth.

5. On a frictionless air table in the lab you measure the velocities of two blocks as $\mathbf{v}_1 = (3\mathbf{i}+3\mathbf{j})\ m/s$, and $\mathbf{v}_2 = (3\mathbf{i}-3\mathbf{j})\ m/s$. The masses of the blocks are $m_1 = 2kg$ and $m_2 = 1kg$. The initial positions of the two blocks are $\mathbf{r}_{1,0} = \mathbf{i}\ m$ and $\mathbf{r}_{2,0} = (\mathbf{i} + 6\mathbf{j})\ m$.

   (a) What are the vector positions of the two particles, $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ as they move across the table?

   (b) What is the position of the center of mass as a function of time, $\mathbf{r}_{cm}(t)$?

   (c) What is the velocity of the center of mass $\mathbf{V}_{cm}$?

   (d) Draw the trajectories of both blocks and of the center of mass.

   (e) What are the velocities of the two blocks with respect to the center of mass:

   $$\mathbf{v}_{1,cm} = \mathbf{v}_1 - \mathbf{V}_{cm} = ?$$

   and similarly for block 2.

   (f) What is the momentum of each block with respect to the center of mass?

   (g) What is the position of each block with respect to the center of mass, as a function of time?

   (h) Draw the trajectories of the blocks as seen from the center of mass.

   (i) Draw the momentum vectors measured by someone moving with the center of mass.

   (j) When will the two blocks collide?

   (k) What is the total kinetic energy?

   (l) The collision is elastic. How do the two blocks move after the collision?

6. In this problem we shall study the motion of two bodies with respect to their center of mass in general terms.

   Take two bodies with masses m_1, m_2, positions $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$, velocities $\mathbf{v}_1$ and $\mathbf{v}_2$.

(a) What is the velocity of each body with respect to the center of mass, using the center of mass velocity given in Equation 9.15?

(b) What is the momentum of each body with respect to the center of mass (*"in the center of mass frame"*)?

(c) What is the total kinetic energy ? Express the total kinetic energy as $\mathrm{KE}_{total} = p_{cm}^2/(2\mu)$, in terms of the momentum p_{cm} of each block, and *"the reduced mass"* μ. Express μ in terms of m_1 and m_2.

(d) How will the two blocks move in the center of mass frame after an elastic collision?

We have found that motion as observed from the center of mass is particularly simple. It is easier to follow interactions and collisions between the parts of a system (the internal motion) from the center of mass frame, and then transform the results to whatever other frame (coordinate system) as needed.

7. You throw a ball of mass $m = 1\,\mathrm{kg}$ with a velocity of $20\,\mathbf{i}\,\mathrm{m/s}$ towards a wall. The ball hits the wall and bounces back with a final velocity of $-10\,\mathbf{i}\,\mathrm{m/s}$.

   (a) What is the change in the ball's momentum, $\Delta\mathbf{p}$? Show also the direction of $\Delta\mathbf{p}$.

   (b) If the collision happens in $0.1\,\mathrm{s}$, find the force, $\mathbf{F}_{\mathrm{wall}}$, exerted by the wall. In which direction is this force acting?

Chapter 10

Energy

Newton's Second Law leads to *conservation laws* under certain conditions. We have already learned about the conservation of momentum which holds if the total (external) force is zero. Other important conservation laws that follow from $\mathbf{F} = m\mathbf{a}$ are the *Law of Conservation of Angular Momentum*, which we will study in Chapter 13, and the conservation law which everyone has heard of, the *Law of Conservation of Energy*, which we will derive from Newton's Second Law, as follows:

Let us take the x component of the equation of motion,

$$F_x = m\frac{dv_x}{dt} \tag{10.1}$$

Multiplying both sides with $dx/dt = v_x$ we find

$$F_x\frac{dx}{dt} = mv_x\frac{dv_x}{dt} \quad \textit{or} \quad F_x dx = mv_x dv_x \tag{10.2}$$

Doing the same to the y and z components of the equation of motion, and adding them up,

$$F_x dx + F_y dy + F_z dz = m(v_x dv_x + v_y dv_y + v_z dv_z) \tag{10.3}$$

This is a relation between the components of the total force acting on the body at each instant, the instantaneous changes (steps)dx, dy, dz in the position, the components of the instantaneous velocity and the instantaneous changes dv_x, dv_y, dv_z in the velocity. These little changes can be added up (integrated) throughout the motion. Integrating both sides of the equation one obtains

$$\int F_x dx + \int F_y dy + \int F_z dz = m\left(\int_{v_{xi}}^{v_{xf}} v_x dv_x + \int_{v_{yi}}^{v_{yf}} v_y dv_y + \int_{v_{zi}}^{v_{zf}} v_z dv_z\right)$$

.

The integral involving velocity components only is easy to evaluate. We obtain:

$$\int [F_x dx + F_y dy + F_z dz] = \frac{1}{2}m[v_x^2 + v_y^2 + v_z^2]_f - \frac{1}{2}m[v_x^2 + v_y^2 + v_z^2]_i. \tag{10.4}$$

The subscripts f and i denote *final* and *initial* conditions, respectively. This can be written as

$$\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = \int \mathbf{F} \cdot \mathbf{dr}, \tag{10.5}$$

where $\mathbf{F}\cdot\mathbf{dr} = [F_x dx + F_y dy + F_z dz]$ is the *inner product* or *dot product* of the force vector $\mathbf{F} = F_x\ \mathbf{i} + F_y\ \mathbf{j} + F_z\ \mathbf{k}$ with the "infinitesimal displacement" or tiny step vector $\mathbf{dr} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}$.

This result is called "***the Work-Energy Theorem***". The combination $mv^2/2$ comes from the $m\mathbf{a}$ side of Newton's Second Law. It is a useful quantity as we will soon find out, so it is given a name: $mv^2/2$ is called the "***kinetic energy***". Kinetic energy is a measure of the motion of the body. Its SI units are mass times velocity squared, $kg\ m^{-2}s^{-2}$. This combination of units is given the special name *Joule*; $1\ J = 1\ kg\ m^{-2}s^{-2}$.

The left hand side of Equation 10.5 is just the difference in the kinetic energy between the start and end of the motion. The right hand side of the equation, $\int \mathbf{F}\cdot\mathbf{dr} \equiv W$, is called the ***work done*** on the body by the total force $\mathbf{F}$. At each step dr of the trajectory the little bit of work done, *the integrand* $\mathbf{F}\cdot\mathbf{dr}$, is the step length dr times the *component* of the force $\mathbf{F}$ *along* the direction of motion $\mathbf{dr}$. The integral of all the bits of work done along the trajectory is the total work done.

According to Equation 10.5, the work-energy theorem, the change in kinetic energy must equal the "work done" on the body by the total force $\mathbf{F}$. If one is not interested in details of the motion, or in what $v(t)$ was and how it changed throughout the motion, then Equation 10.5 gives a shortcut to calculate the final speed given the initial speed, provided one can evaluate the integral for the work done.

For some special forces, called "*conservative*" forces, the work integral is easy: it turns out to be just some function $U(x, y, z)$ *of coordinates alone*, so that the work done in going from initial point (x_1, y_1, z_1) to final point (x_2, y_2, z_2) *does not depend on the path or the details of the motion*. The work done by conservative forces depends only on the value of this function U at the initial and final points:

$$\text{Work} = \int \mathbf{F}\cdot\mathbf{dr} = -\int dU = U(x_1, y_1, z_1) - U(x_2, y_2, z_2) \quad \text{(for conservative forces)} \tag{10.6}$$

For this simple way to calculate work to be valid, the integrand $\mathbf{F}\cdot\mathbf{dr}$, the little bit of work done at every step, must be equal to the infinitesimal difference dU in taking this step. The definition for U is actually

$$\mathbf{F}\cdot\mathbf{dr} = -dU. \tag{10.7}$$

The minus sign is introduced here simply in order not to have any minus signs in the conventional statement of the final result below. Now such an infinitesimal change dU in taking the step $\mathbf{dr} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k} = (dx, dy, dz)$ is the combined change in U resulting from all three components dx, dy, dz of the step $\mathbf{dr}$:

$$dU = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial u}{\partial z}dz \tag{10.8}$$

The "*partial derivative*" $\partial U/\partial x$ means the derivative of U if only x changed and y and z were constant. Thus the total change dU is the sum of the changes due to the steps dx in the x direction, dy in the y direction and dz in the z direction.

Now the question is, for a given force $\mathbf{F}$, whether the work done by that force can be calculated easily by

$$\mathbf{F}\cdot\mathbf{dr} = -\,dU. \tag{10.9}$$

By writing out both sides of the equation,

$$F_x dx + F_y dy + F_z dz = -\,(\frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial u}{\partial z}dz) \tag{10.10}$$

The force must satisfy, at all points in space,

$$\mathbf{F} = (F_x, F_y, F_z) = -\left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z}\right) \tag{10.11}$$

Each component of the force vector, as a function of coordinates, must be (minus) the corresponding partial derivative of this single function $U(x, y, z)$. The vector made of the *partial derivatives* of a single scalar function is called the *gradient* of that function, and denoted as

$$\nabla U \equiv \left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z}\right) = \frac{\partial U}{\partial x}\,\mathbf{i} + \frac{\partial U}{\partial y}\,\mathbf{j} + \frac{\partial U}{\partial z}\,\mathbf{k} \tag{10.12}$$

Thus, for the work integral to be easy and determined only by the endpoints of the motion, the force must have the property $\mathbf{F} = -\nabla U$. *Forces having this property are called conservative forces.* This is a very special property. Not all forces in nature have this property. But fortunately many important fundamental forces are conservative, including the gravitational force and the electrostatic (Coulomb) force responsible for making atoms and binding matter. (We will learn about the Coulomb force and the structure of atoms in later chapters.) The function U is called the "*potential energy*". If there are only conservative forces doing work, the work energy theorem becomes:

$$\begin{aligned} \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 &= \int \mathbf{F}\cdot\mathbf{dr} \\ &= -\int dU = -(U_f - U_i) = U_i - U_f \end{aligned} \tag{10.13}$$

so that

$$\frac{1}{2}mv_f^2 + U_f = \frac{1}{2}mv_i^2 + U_i = constant \equiv Energy \tag{10.14}$$

This is the *Law of Conservation of Energy.* We now see that the minus sign in the definition of potential energy was introduced so that the final conservation law above does not have any minuses in it. In situations where the total mechanical energy (the sum of the kinetic and potential energies) is conserved, one can get information on the final positions and velocities from the initial conditions, in terms of the one *constant of the motion*, energy. When there are some conservative and some non-conservative forces, the Law of Conservation of Energy becomes:

$$\frac{1}{2}mv_f^2 + U_f^{con} = \frac{1}{2}mv_i^2 + U_i^{con} + \int \mathbf{F}_{noncon}\cdot\mathbf{dr} \tag{10.15}$$

where U^{con} is the total potential energy due to all the conservative forces and $\mathbf{F}_{noncon}$ is the sum of all the non-conservative forces.

Two examples of conservative forces are the universal gravitational force and the Coulomb force, which both have the form

$$\mathbf{F} \propto r^{-2}\hat{\mathbf{e}}_r$$

where r is distance from the center of the force (from the other body that applies the force), and $\hat{\mathbf{e}}_r$ is the unit vector in the r direction. Take the gravitational force on a body with mass m, applied by another body, of mass M, as a function of position $\mathbf{r} = r\hat{\mathbf{e}}_r$ from the center of the body with mass M:

$$F = -\left(\frac{GMm}{r^2}\right)\hat{\mathbf{e}}_r = -\frac{\partial}{\partial r}\left(-\frac{GMm}{r}\right)\hat{\mathbf{e}}_r = -\nabla U(r) \tag{10.16}$$

The force is a function of r alone, so the potential energy must also be a function of r alone and the partial derivative of the potential energy in the $\hat{\mathbf{e}}_r$ direction gives the force. You can get the same result with a longer calculation if you blindly take partial derivatives working in rectangular coordinates (x, y, z) with $r^2 = (x^2 + y^2 + z^2)$ (see the Appendix at the end of this chapter).

In motions near the surface of the Earth the gravitational force is constant to a good approximation, because the distance of the moving body from the center of the Earth remains almost constant, equal to the radius of the Earth (see Problems 1 and 2 at the end of Chapter 2). The local direction of the radius r from the center of the Earth does not change during small motions near the surface of the Earth. Take that direction to be the upward direction, call it the z direction, with the unit vector $\hat{\mathbf{k}}$. The force is then approximately:

$$\mathbf{F} = -\frac{GM_\oplus m}{R_\oplus^2}\,\hat{\mathbf{k}} = -mg\,\hat{\mathbf{k}} = -\frac{\partial}{\partial z}(mgz)\,\hat{\mathbf{k}} = -\nabla U(z) \tag{10.17}$$

where $M_\oplus$ and $R_\oplus$ are the mass and radius of the Earth respectively, and g is the gravitational acceleration on the surface of the Earth. The sign $\oplus$ denotes the Earth. The potential energy function becomes the familiar

$$U(z) = mgz \tag{10.18}$$

This is *only the approximate form of the potential energy associated with one particular force, gravity, valid only near the surface of the Earth.*

10.1 Non-Conservative Forces

Friction is the most important and common kind of non-conservative force. It may seem that in one dimension all forces should be conservative, since in a one dimensional motion $F_x(x) = -\partial U/\partial x$ must surely have a solution: It seems any function $F_x(x)$ of position must be the derivative of something, at first one thinks a potential energy function surely can be found such that minus the derivative of the potential energy gives the force. But friction is *not even a function* of position: it is not determined only by position, by where the object is; friction also depends on velocity; even in one dimension friction depends on which way the object is moving. At any particular point x, the friction force must always be opposite to the direction of motion (or for static friction, opposite to the direction of the tendency to move, as a result of all the other forces), so the friction force acts to the right if the motion is towards the left, and it acts to the left if the motion is towards the right. The direction of the friction force depends on the motion, and not only on the coordinate x. If there were some potential energy $U_{fric}(x)$ to correspond to the friction force, what would be the slope (= gradient) of that $U_{fric}(x)$?! So even in one dimension, friction (and all forces that depend on direction and/or magnitude of *velocity*) cannot be conservative.

The work done by friction is always negative because a force of friction $\mathbf{F}_f$ is always opposite in direction to $\mathbf{dr}$. The work done by friction produces heat. It turns out that at a microscopic level, when friction forces are expressed in terms of the fundamental forces, these fundamental forces are conservative, so the work done does go into changing the kinetic and potential energies in microscopic motions of the parts of the system. At a microscopic level, energy is conserved by the fundamental interactions. From a macroscopic point of view one identifies the large scale conservative forces, like gravity, while the effect of detailed microscopic forces is not resolved. These are lumped together as non-conservative forces, like friction, doing their work to transfer energy into random microscopic channels – that transfer of energy is called the generation of *heat*.

10.2 Potential Energy

A simple example of a *potential energy surface* is a landscape. The gravitational potential energy near the surface of the Earth is just proportional to height. Look at a landscape with hills and valleys. The top of a hill has higher potential energy than the bottom of the next valley. The height of each point on the landscape of the earth, plotted against the longitude and latitude, (the vertical or z coordinate as a function of the "horizontal" x and y coordinates) is a surface. This is just the surface of the real landscape on the Earth. The landscape or the potential energy surface determines motions like the flow of rivers or the rolling of a stone down a hillside.

In many problems in science the effect of forces can be described in terms of potential energy surfaces. The slope of the potential energy surface gives the direction and strength of the force. Not all forces can be described in terms of a potential energy. Friction forces cannot be described in terms of potential energy. But many important forces can be described with potential energy surfaces. Gravity is one example. Another important example is the electrostatic force. When we discuss electromagnetism we shall see that *voltage*, which is defined as the electrostatic potential energy of a unit charge, is a very important quantity. In a situation with charged particles and electrostatic forces it is useful to know what the voltage is at each point in space. If the charges have to move about in a plane you can make a potential energy surface by plotting the voltage as the z coordinate at each point $(x,\ y)$ in the plane. If you want to plot the voltage for a three dimensional problem, for each point $(x,\ y,\ z)$, then you cannot make a picture or model of the potential energy surface: we are in a three dimensional world and we cannot visualize or plot, or make a model, of the voltage as the fourth dimension, against the three coordinates $(x,\ y,\ z)$. But the mathematical properties of the voltage "surface" as a function of $(x,\ y,\ z)$ are similar to the landscapes we can see.

The potential energy in many problems depends on coordinates other than just the position of some object. The potential energy might depend on the distances between each pair of atoms in a molecule with say 25 atoms. So the potential energy would depend on 300 different distances.

Question: Where does the number 300 come from?

In addition, the potential energy might depend on the angles between the lines joining the atoms. So there are lots and lots of coordinates. Each set of values of 300 (or 560 or whatever number of) coordinates is a point in a 300 dimensional space. There is a value of the potential energy U for this point. Plotting the value of U against the values of the 300 coordinates gives a surface in a 301 dimensional space. A lot of the mathematics of dealing with potential energy surfaces is similar no matter what the dimension (number of coordinates) in the problem is. We can see what kinds of interesting things can happen by studying what kinds of structure occur on a landscape. For this we shall first study the simplest kind of potential energy landscape: there is only one coordinate in the problem. The potential energy depends on the position of a particle, say, along the x axis. Then we can make pictures and graphs. A two dimensional graph in your notebook or computer screen will show the potential energy $U(x)$ on the vertical axis as a function of x. This could be the cross section across some hills and valleys in a real landscape in the case of gravitational potential energy near the surface of the Earth: In this special case the potential energy $U(x)$ is just proportional to the altitude z, $U(x) \cong mgz$, so a cut through the landscape, $z(x)$ or the whole landscape $z(x, y)$ is actually a picture of the potential energy $U(x)$ or $U(x, y)$.

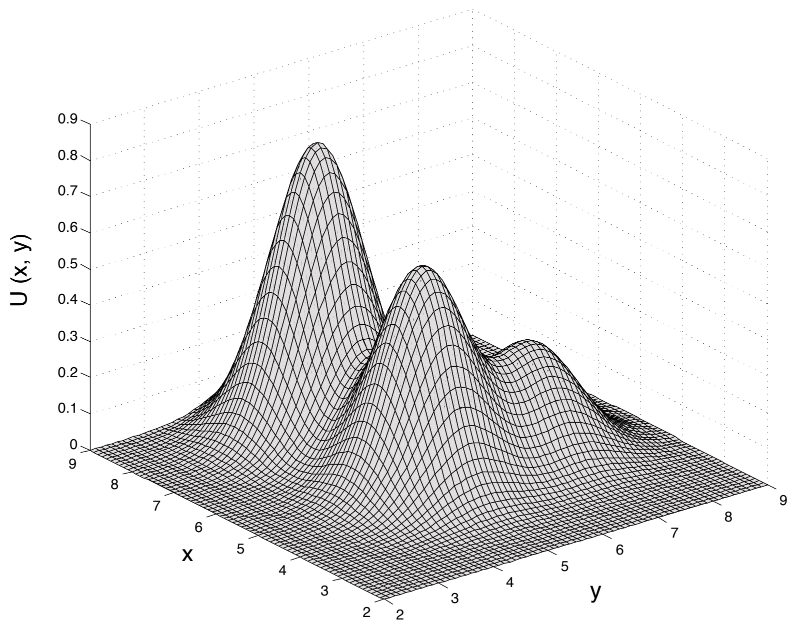


Figure 10.1:

> **Exercise:** Give an example of a real problem for which these pictures are useful!

A graph or a surface will have some *minimum* points. These are places with less potential energy than any of the surrounding points. If a particle is placed at such a point, and it is at rest, then it will just stay there, at rest. It has no kinetic energy. All it has is potential energy. That is its total energy. Being at a *minimum* of potential energy, the energy is not enough to move the particle to any other point around!

All this potential energy description is just a way to describe the possible effects of forces on the system. Consider the simplest system, a particle moving in one dimension. According to Newton's Second Law, the particle at rest *remains* at rest when there are no net forces on

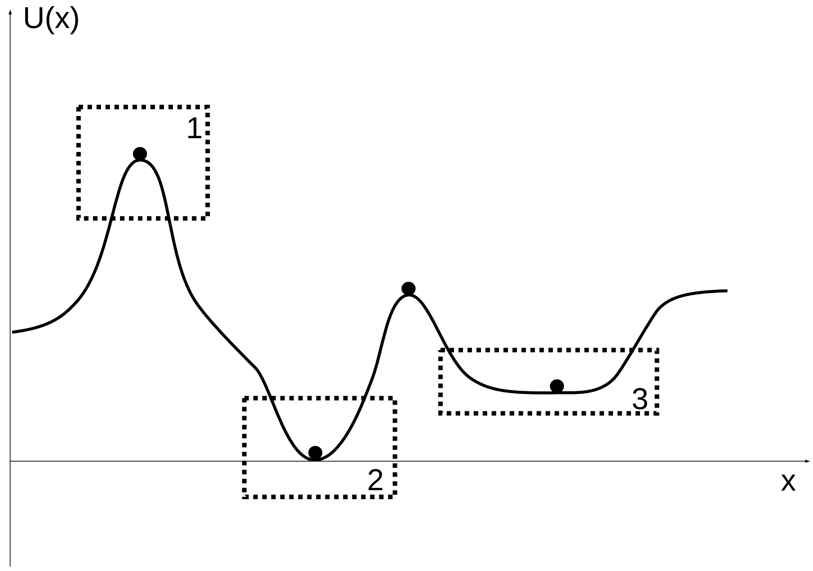


Figure 10.2:

it. The particle remains at rest at the point where the potential energy is minimum. So this point of minimum potential energy is a special point where there are no net forces on the particle.

The minima of potential energy surfaces correspond to zero net force. A point of minimum potential energy is called an *equilibrium point.* If the particle is placed at a point near the minimum of the potential energy, at a point near the bottom of the valley but not right at the bottom, and initially at rest, what will happen?

Place the particle at the equilibrium point in Frame 2 of Figure 10.2. It is not going to move when you release it. If you now place the particle at any point *near* the minimum it will start from rest, and move towards the equilibrium point.[1] Near the minimum, forces are *towards* the minimum (equilibrium) point! A particle starting at rest is accelerated towards the equilibrium point. It gains speed. When it reaches the equilibrium point it is already moving at some speed so it moves *through* the equilibrium point. It goes to the other side. Now it is moving "uphill". The forces are still towards the equilibrium point, so now they are directed "opposite" to the velocity of the particle. The forces are now decelerating the particle. It will reach a point where its velocity is zero. It stops there momentarily.

Remember, the forces are acting *towards the equilibrium point all the time*, so now the particle is accelerated back down the hill towards the equilibrium point. It gains speed, passes the equilibrium point, moves up the hill towards the point it started from, stops there momentarily and repeats the whole thing again.

Let us look at this from the point of view of energy. Initially the particle had no kinetic energy. It was higher up the hill then the equilibrium point, so it had some potential energy *with respect to the equilibrium point.* Let us take the value of the potential energy at the equilibrium point to be zero. Then the potential energy at the initial position was higher, it

[1]This simulation is available in NS 101- SUCourse. In the simulation zoom up the yellow frame and place the particle anywhere. Then click play to watch the particle's motion.

had a positive value. The particle moves down and passes through the equilibrium point. At the moment when it is passing through the equilibrium point, it has only kinetic energy. When it moves up the hill on the other side, it will move up to a point where all of its total energy is potential, there is no kinetic energy. This is a point of momentary rest. The instantaneous speed is zero. Well, if the potential energy is just proportional to height, as for gravitational potential energy, then it will move up to the same height that it started from. If the shape of the potential energy surface is not symmetric about the bottom, the distance of the turning point from equilibrium will be different from the distance of the starting point from equilibrium. (See Figure 10.2.) In any case the particle will go back and forth moving through and near the equilibrium point, never getting further than the starting and turning points.

Question: If you add some constant to the potential energy, so that the minimum value is not zero but some other number, say 5 m or -3 m, correspondingly raising or lowering the entire landscape, does this change the forces at each point and the location of equilibrium points?

Motions starting in the neighborhood of a minimum potential energy point always remain in the neighborhood. That is what their available energy, determined by the starting conditions, allows. So points of minimum potential energy are called "*stable equilibrium points*".

Now consider motion in the same potential energy landscape, but with friction forces also present.[2] Start the particle from some point near the equilibrium point. This time the particle gains speed every time it goes downhill: the gravity potential always accelerates the particle towards the stable equilibrium point. But the friction force is against the direction of motion of the particle no matter in what direction it moves. The friction force has no stable equilibrium point or any other special points it cares about. The friction force depends on velocity, always acting to brake the motion. The net effect is that the particle moves back and forth a number of times, depending on the initial conditions. The starting point and the initial velocity are the initial conditions. These conditions define the total mechanical energy available initially. The particle is losing energy to friction all the time. The total potential + kinetic *mechanical* energy is *not* conserved. So each time the particle climbs the hill it climbs to a lesser height. Eventually it stops right at the equilibrium point and stays there.

Question: Why does the particle stop at the equilibrium point and not somewhere else?

Recall that the potential energy could be a function of coordinates like distances between atoms, angles between the chemical bonds of neighboring atoms, and all kinds of parameters which define forces and potential energies. This is the analogue of the particle sitting at rest at the bottom of a valley. Now change the configuration a little bit, the molecule was distorted, say by a water molecule passing by. It now oscillates about (through) the stable equilibrium configuration. This configuration is the analogue of the particle released from the hillside and oscillating about the bottom.

[2] In the simulation in NS 101- SUCourse, it is possible to observe the particle's motion near the equilibrium point with and without friction

Let us return to the simple example with the particle. What will happen near the top of a hill, as in Frame 1 in Figure 10.2? Place the particle right at the summit, at rest. It just stays there. So there is no net force at the summit, the point where potential energy is maximum. This is an *equilibrium point*. Now place the particle at a point near the summit. The particle moves *down* the slope, which means *away* from the maximum of potential energy. So a point where *potential energy is maximum* is an *unstable equilibrium* point. If placed exactly at this point a particle stays there. If displaced a little bit, it will move even further. It moves further and is accelerated away from the maximum point, so it moves further faster and faster and just runs away wildly.

This situation is a simple example of *an instability*: A small initial disturbance (moving a bit away from the summit) leads to more, larger displacements and ever increasing speeds. This is also called *positive feedback*: The system responds to an initial condition by making more of the same, and *amplifying* it. (The oscillations about a stable equilibrium point, with the force always pointing back towards the equilibrium point exemplify *negative feedback*).

When the particle is started from rest near the summit, its total energy is just the initial potential energy. This is more than the potential energy at the points downhill, so as the particle moves down the slope away from the summit it will have more and more kinetic energy. Turning on the friction will make the particle move away at slower rates, since some of the energy is now dissipated (turned into heat), it does not all go to kinetic energy. But the particle is not going to be stopped by friction somewhere on the slope. If the particle came to rest momentarily, gravity would start to move it downward. If the slope had no bottom, the particle would just keep moving away whether there is friction or not. However in a real landscape every downward hillside slope eventually joins a valley or a plain. Similarly in most real potential energy surfaces describing situations in nature no valley is surrounded by infinitely high slopes and no summit is surrounded by bottomless falls all around. In potential energy surfaces, there are the minima, the stable equilibrium points, there are the *maxima*, which are *unstable* equilibrium points, and there can be flat portions.

Let us now look at a plain section of the landscape, like that in Frame 3 in Figure 10.2. First let us consider what happens when there is no friction present. If the particle is released from rest at a point on the slope on one side of the plane, it will accelerate down the hill, and move across the plane at constant velocity, since there are no horizontal forces on the flat horizontal plane. Initially all the energy was potential energy. Moving into and through the plane all of this energy is now kinetic energy. At the other end of the plane the particle enters a slope. It is going uphill, so the force of gravity is against the direction of motion. Gravity is decelerating (braking) the particle's motion. It will climb up to the same height as its starting point. Here the potential energy is the same as the initial potential energy, which is just the total energy. So at the turning point, which is at the same height as the starting point, there is no kinetic energy. The slopes on the two sides of the plane are not symmetrical, so the turning point is not at the same horizontal distance from the plane as the starting point, though they are at the same height. The particle is momentarily at rest at the turning point. This is the point at which gravity has finally stopped the upward motion. Under gravity, the particle now starts to move down the slope, back into the plane where its energy is all kinetic, across the plane at constant velocity, and back up the slope to its starting point. Then the whole motion starts again. Whether there is a single point or an extended plane at the bottom of the valley (the potential energy minimum), the motion around it is periodic, repeatedly moving through the equilibrium plane, but never getting far, returning all the time because of the finite energy available.

Now put friction in the plane. Every time the particle moves through the plane some of the kinetic energy will be dissipated. So it moves up to a lower turning point at each encounter with one of the slopes. Eventually it will stop. It has to stop in the plane. Where did the energy go?

These are the only possibilities in the one dimensional problems with potential energy given by a curve $U(x)$.

Question: Can you see another distinct possibility, another kind of special point, in higher dimensional situations? What is a *saddle point?* Make a picture.

10.3 Elastic Collisions

Now that we have learned about kinetic and potential energy, and conservation of energy, we can take another look at collisions and interactions. Two bodies may deform each other during an interaction. This can happen even when the two bodies do not actually touch - consider the tides that the Moon induces on the Earth's oceans, and the effect of the Sun on a comet. The deformations and resulting motions within the body, for example the ebb and flow of tides and currents in the ocean will encounter frictional forces, leading to the transformation of some mechanical energy into heat. In collisions or in the continuous interactions between bodies in a bound system (like the Earth-Moon system) some mechanical energy is lost to heat and the total mechanical energy is not conserved. However in many collisions the amount of energy lost to heat is negligible, and the total mechanical energy is conserved to a very good approximation. Such collisions are called *elastic collisions.*

In the initial and final states of the collision the two particles are very far from each other and the potential energy of their interaction is the same in the final state as in the initial state, usually taken to be zero. In an elastic collision, the total mechanical energy is the same in the initial and final states, to a good approximation, and the potential energy is also the same, therefore the initial and final total kinetic energies are also equal to a good approximation. The total kinetic energy is a conserved quantity, in addition to the total momentum which is conserved in the absence of external forces. For an elastic collision the conservation laws give

$$\mathbf{p}_{1,i} + \mathbf{p}_{2,i} = \mathbf{p}_{1,f} + \mathbf{p}_{2,f} \quad \text{Conservation of Momentum} \tag{10.19}$$

$$\frac{{p_{1,i}}^2}{2m_{1,i}} + \frac{{p_{2,i}}^2}{2m_{2,i}} = \frac{{p_{1,f}}^2}{2m_{1,f}} + \frac{{p_{2,f}}^2}{2m_{2,f}} \quad \text{Conservation of Energy.} \tag{10.20}$$

For most forces, including electrostatic and gravitational forces, a property called *angular momentum* is also conserved as we will discover in the next chapter. The conservation of angular momentum requires that the final momentum vectors $\mathbf{p}_{1,f}$ and $\mathbf{p}_{2,f}$ lie in the same plane as the initial momentum vectors $\mathbf{p}_{1,i}$ and $\mathbf{p}_{2,i}$. Choosing our coordinates in that plane, the equation for conservation of momentum, Equation 10.19, a vector equation, gives two equations, one for each component of the momenta. Together with the conservation of (kinetic) energy in an elastic collision, Equation 10.20, we have three equations relating the final momenta to the initial momenta.

CHAPTER 10 - PROBLEMS:

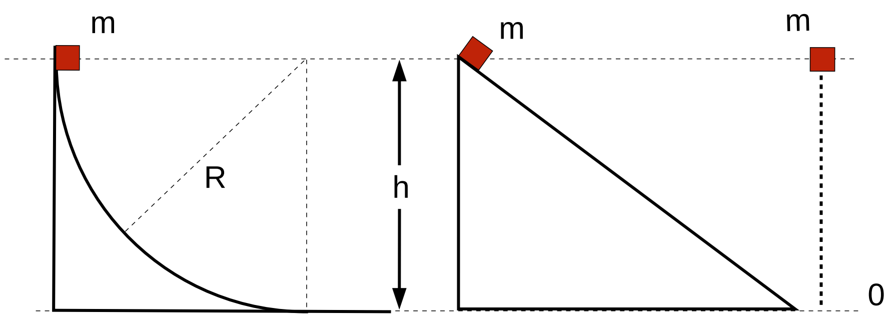


Figure 10.3:

1. The three tracks in Figure 10.3 are frictionless. Three identical blocks are released from rest from the same height, one on each track. What are their speeds when they reach the ground?

2. Suppose there is friction on the first two tracks in Problem 1 and the coefficient of friction is the same, compare the speeds of the three blocks when they reach the ground. The first track is longer than the second track.

3. What is the work done by friction on a block of mass M which starts from rest at the top and moves down an inclined plane of height h, inclination angle θ and coefficient of kinetic friction μ. What is the kinetic energy of the block when it reaches the bottom of the inclined plane?

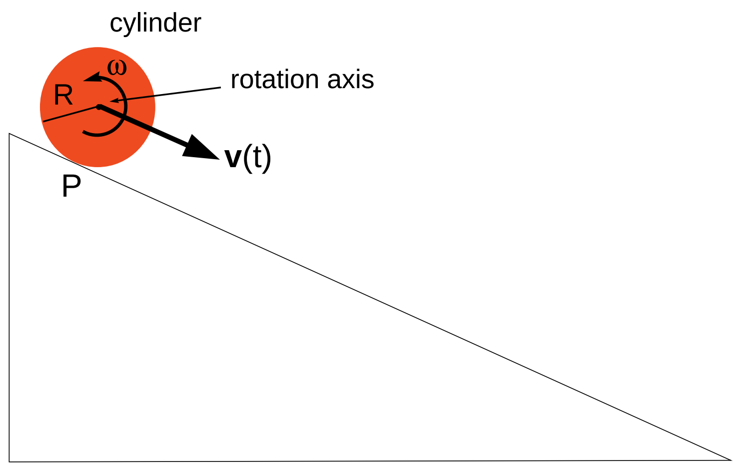


Figure 10.4:

4. **Kinetic Energy of a Rolling Body**: A cylinder on an inclined plane can slide without rolling at all. In that case the kinetic energy is just

$$\mathrm{KE} = \frac{1}{2}mv^2$$

If the cylinder does not slide at all but only rotates down the inclined plane then the contact point P with the inclined plane is momentarily at rest with respect to the inclined plane at every instant. The axis of rotation of the cylinder moves with velocity $v(t)$ parallel to the surface of the inclined plane. When there is no sliding, the velocity of the point P is $v(t) = \omega(t)R$ where ω is the angular speed of the cylinder. We have $v(t) - \omega(t)R = 0$, so that $\omega(t) = v(t)/R$.

If there is sliding as well as rotation, then the contact point P moves with the sliding velocity $V(t)$. Then $v(t) = \omega(t)R + V(t)$ and $\omega(t) = [v(t) - V(t)]/R$.

For a rotating body,in addition to $mv^2/2$, there is an extra rotational kinetic energy, of the different parts of the body moving in rotation with respect to the axis. Now let us calculate this energy.

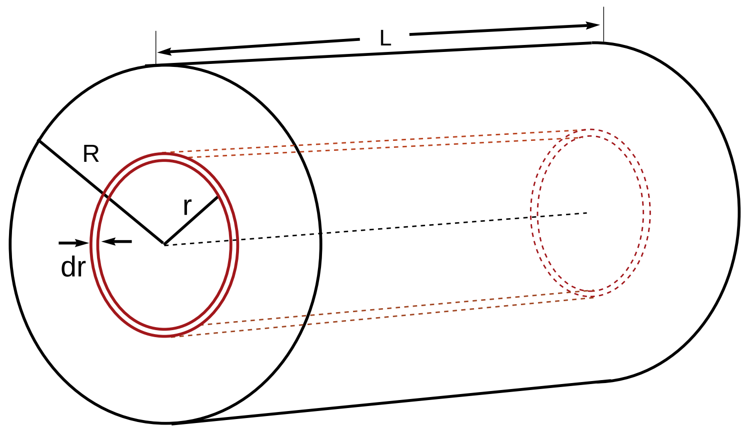


Figure 10.5:

Take a rigid cylinder of radius R and length L and mass M as seen in Figure 10.5.

If we consider a cylindrical shell of thickness dr and radius r,

(a) What is the velocity of a point on this cylindrical shell with radius r, if the cylinder is rotating around the axis passing through its center with an angular velocity ω?

(b) If the cylinder has a uniform density ρ, what is the mass dm of the shell with radius r and thickness dr? Note: dm is equal to the density times the volume of the thin cylindrical shell.

(c) What is the rotational kinetic energy of the cylindrical shell of mass dm at distance r from the axis?

(d) What is the rotational kinetic energy of the entire cylinder? Note: From the previous steps we figure out the rotational kinetic energy of the thin cylindrical shell. Now we need to add the kinetic energies of all the similar cylindrical shells with radius from $r = 0$ to $r = R$, thus we need to integrate over r.

(e) The total rotational kinetic energy is proportional to ω^2. One can write it as:

$$\text{KE}_{rot} = \frac{1}{2}\, I\omega^2$$

Here I is called the "moment of inertia" of the body (around that particular axis of rotation.

What are the dimensions of moment of inertia?

What is the moment of inertia of the cylinder in Figure 10.5

(f) Suppose we have a body of uniform density and arbitrary shape rotating about an axis. Divide the body into coaxial thin cylindrical cells of thickness dr, distance r from the axis, height $h(r)$ parallel to the axis.

How would you calculate the rotational kinetic energy?

How would you calculate the moment of inertia independently?

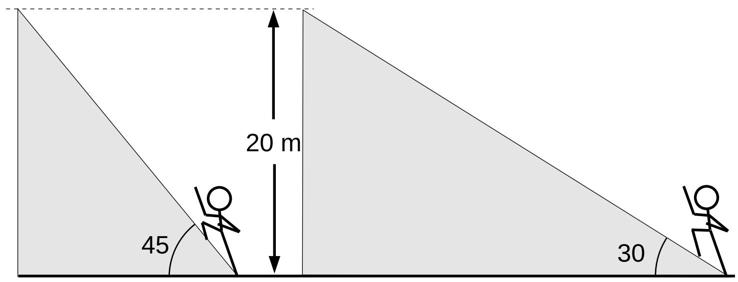


Figure 10.6:

5. (a) Calculate the work done on you by the Earth's gravitational force as you run up each of the tracks in Figure 10.6. Use your own mass and take $g \cong 10\ m/s^2$ in your calculations.
   (b) How much work have you done on the Earth in each case?
   (c) What is the difference between your final potential energy, on top of the track, and your initial potential energy, at the bottom, in each case.

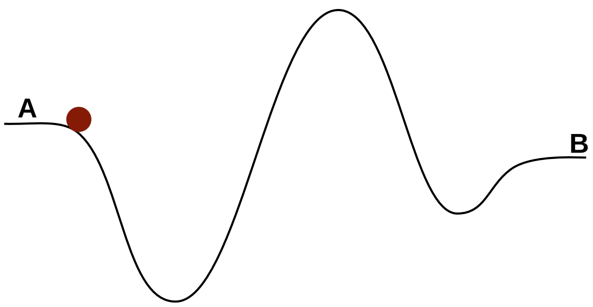


Figure 10.7:

6. A ball follows the track shown in figure 10.7.
   (a) What are the equilibrium points on this track?

(b) Are any of these stable equilibrium points? Which one(s)?

(c) If the ball is released from point A initially and it rolls without slipping on the track, how can you describe its subsequent motion? (See Problem 10.4.)

(d) When does the ball have maximum kinetic energy?

(e) If there is some slipping on the track how can you describe its subsequent motion after the ball is released from point A?

7. A curved track is placed on a table of height H as shown in Figure 10.8. A mass M_1 is released from the top of the frictionless track. Another mass $m_2 = M_1/2$ is initially at rest on the table, at the bottom of the track. When M_1 reaches the bottom of the track, the masses collide elastically, and subsequently they both fall to the ground.

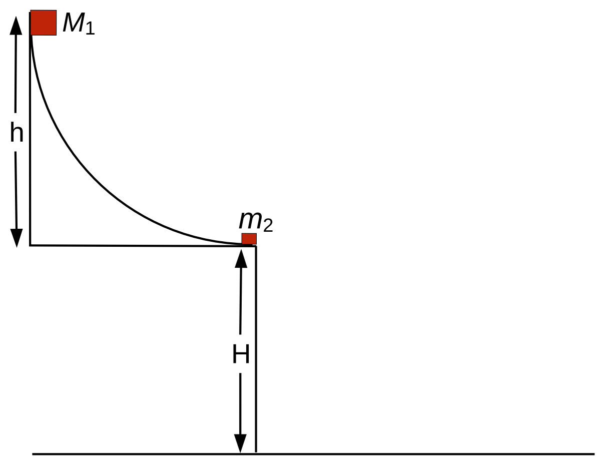


Figure 10.8:

(a) Find the velocity of M_1 when it reaches the bottom of the track.

(b) Find the velocities of M_1 and m_2 after the collision.

(c) Find the distance from the edge of the table that each mass travels before they fall on the ground.

APPENDIX:

Calculation of the gravitational acceleration in cartesian coordinates:
The gravitational potential energy in cartesian coordinates $(x,\ y,\ z)$ is:

$$U(r) = \frac{GMm}{\sqrt{x^2+y^2+z^2}}$$

Gravitational force is the gradient of the potential:

$$\mathbf{F} = -\frac{\partial U}{\partial x}\mathbf{i} - \frac{\partial U}{\partial y}\mathbf{j} - \frac{\partial U}{\partial z}\mathbf{k}$$

Taking the derivatives we obtain:

$$\mathbf{F} = -\frac{GMm}{(x^2+y^2+z^2)^{3/2}}(x\ \mathbf{i} + y\ \mathbf{j} + z\ \mathbf{k}) \tag{10.21}$$

The unit vector $\mathbf{e}_r$ in the direction of the vector $\mathbf{r}$ is simply $\mathbf{r}$ divided by its magnitude r:

$$\mathbf{e_r} = \frac{(x\ \mathbf{i} + y\ \mathbf{j} + z\ \mathbf{k})}{\sqrt{x^2+y^2+z^2}} \tag{10.22}$$

Comparing equations (10.21) and (10.22), we obtain the equation for the gravitational force:

$$\mathbf{F} = -\frac{GMm}{r^2}\mathbf{e_r}. \tag{10.23}$$

Using the Newton's 2nd law ($\mathbf{F} = m\mathbf{a}$), we find the acceleration:

$$\mathbf{a} = -\frac{GM}{r^2}\mathbf{e_r}. \tag{10.24}$$

Chapter 11

* Damping and the Exponential Function

11.1 Damping Forces Proportional to Velocity

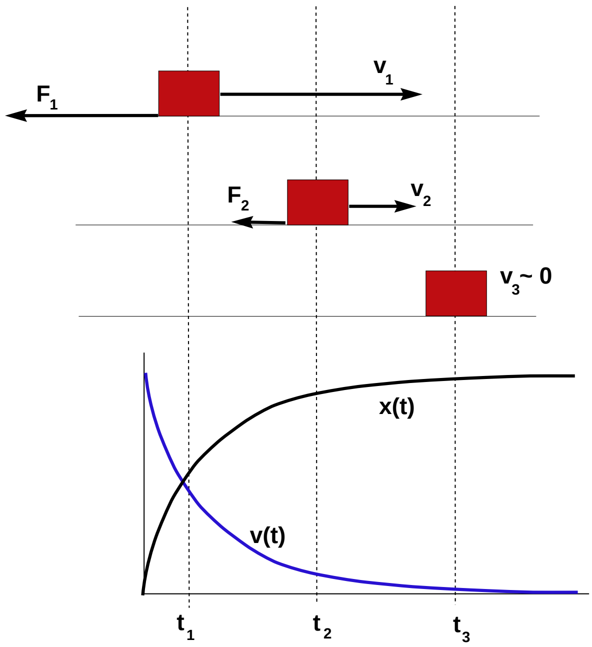


Figure 11.1:

In Figure 11.1 the block is moving to the right and there is a damping force in the form

$$\mathbf{F} = -\alpha \mathbf{v}, \tag{11.1}$$

where α is a positive constant. The damping force is in the opposite direction to the velocity and it is proportional to the magnitude of the velocity. This is a good approximation in many situations with damping.

Simulation: To follow motion under the effect of a damping force of this form, check out the simulation in NS 101 SUCourse under "Damping". In this simulation, the values of initial velocity, mass and damping coefficient are restricted as follows:

$$0 < v \leq 500\ m/s, \qquad 0 < m \leq 10\ kg \qquad 0 \leq \alpha \leq 10\ Ns/m$$

First consider $\alpha = 0$. This means $\mathbf{F} = 0$. How does the block move when there is no force ?
Now make $\alpha = 1$. Watch the motion of the block and view the graphs of the position $x(t)$, the velocity $v(t)$, and the force $F(t)$.

Let us work out how this motion takes place, using the damping force in Equation 11.1 and Newton's 2nd Law, for a particle of mass m.

$$\begin{aligned} m\frac{d\mathbf{v}}{dt} &= \mathbf{F} = -\alpha\mathbf{v} \\ \frac{d\mathbf{v}}{dt} &= -\frac{\alpha}{m}\mathbf{v} \end{aligned} \tag{11.2}$$

The rate of change of velocity vector at any moment in time is just a constant times the velocity at that moment! As the change in velocity lies on the same line as the velocity itself, the motion is one dimensional, keeping to the direction of the initial velocity. Let us rewrite the above expression with a new constant τ,

$$\frac{d\mathbf{v}}{dt} = -\frac{\mathbf{v}}{\tau}. \tag{11.3}$$

The constant $\tau \equiv m/\alpha$ has dimensions of time.

This kind of equation, stating that the derivative of some quantity is just proportional to that quantity is very common, and very important in science. It is so important that we shall devote the next section to the solution of equations like Equation 11.3. The solution for this kind of change involves the *exponential function*.

11.2 The Exponential Function

An example with dynamics mathematically similar to the case of damping proportional to velocity is the problem of unchecked population growth in biology. In the unrealistic case of no deaths, the rate of new births in a population of living organisms will be some constant times the current population. The meaning of the constant is simply the fraction of the population that give birth in a time interval dt. Consider a human population of N individuals. The number of babies born in a particular year is proportional to the number of women of child bearing age in the population, which in turn is proportional to the total population N. The rate of population growth dN/dt, in units of the number of babies born per year, is therefore just proportional to the population N:

$$\frac{dN}{dt} = C\ N. \tag{11.4}$$

This equation is of the same form as the equation of motion for a particle damped by a damping force linear in velocity, Equation 11.3. The only difference is in the sign of the constant.

Another important example concerns decay reactions. In a chemical reaction or nuclear decay like $A \to B + C$, the rate of atoms of species A decomposing or nuclei decaying is proportional to the number available to decay or decompose,

$$\frac{dN_A}{dt} = -K\ N_A. \tag{11.5}$$

In this case the proportionality constant is negative since the number N_A of atoms of species A is decreasing.

To understand the solution of all equations of this type, let us consider the simplest case,

$$\frac{dN}{dt} = N \tag{11.6}$$

When we understand that, we can easily understand the general case $dN/dt = CN$, where C is a positive or negative constant.

At $t = 0$, we have $N(0)$ of these things, whatever they are. We know that their number changes according to the equation $dN = Ndt$. After every second, or every time unit, whatever units we are using, there are N more things! So after dt seconds , where dt is a very short time, much shorter than the unit time, we already have an increase $dN = Ndt$. We want to find N at any later time t : We want to find $N(t)$.

Let us do this step by step. Say, we do this in n steps, so we divide the time t into n time steps such that $t = n\ dt$. After the first little interval of time the number we have is what we started with, $N(0)$, plus the increase, which itself is proportional to $N(0)$, so:

$$N(dt) = N(0) + dN = N(0) + N(0)dt = N(0)[1 + dt] \tag{11.7}$$

Now let us see what we get after the next time step dt. We want to find $N(2\ dt)$:

$$\begin{aligned} N(2\ dt) = N(dt) + dN &= N(dt) + N(dt)\ dt \\ &= N(dt)[1 + dt] = N(0)[1 + dt]^2. \end{aligned} \tag{11.8}$$

And after the next step,

$$N(3\ dt) = N(2\ dt)[1 + dt] = N(0)[1 + dt]^3. \tag{11.9}$$

We get to time $t = n\ dt$ after n steps, and find

$$N(t) = N(n\ dt) = N(0)[1 + dt]^n = N(0)[1 + t/n]^n. \tag{11.10}$$

So we know how to calculate $N(t)$ starting from $N(0)$. Do this approximate calculation yourself: see Problem 1 at the end of this chapter.

This is a unique prescription. Start from $N(0)$, and apply Equations 11.6 and 11.7, that the rate of increase is just proportional to the present value of N at each step, and we will get $N(t)$. But the curve for $N(t)$ in this figure is jerky. This is because we decided to calculate in n steps. This is an approximate calculation! The only way to calculate anything numerically, on a pocket calculator or a computer is to calculate it approximately, to some degree of accuracy. How many steps n do you want to go through to get to some fixed t? To find, for example, the population of bacteria in your dish, in an unchecked population increase (no deaths!) through one day, you could calculate in one hour steps. That would take 24 steps. Or you could take it in steps of seconds. That would take 86400 steps. The curve would be more continuous – less jerky, and the result would be more accurate in the latter case. So the more steps you employ, dividing your time t into the n steps, and making n larger and larger, dt smaller and smaller, a nicer smoother curve you get:

$$N(t) = N(0) \lim_{n\to\infty} [1 + (t/n)]^n. \tag{11.11}$$

Note that this curve is unique. There is only one way to get from $N(0)$ to $N(t)$. If we start with a different $N(0)$, say $N_{new}(0) = 2.53\, N(0)$, we just multiply the whole curve with 2.53. The scale is set by $N(0)$. The shape of the curve does not depend on $N(0)$! Just like there is only one shape of a circle, or parabola, or equilateral triangle, although these shapes come in different sizes, there is only one shape of the curve describing how something grows if its rate of growth is just proportional to its value at every point. The unique shape is given by the function

$$f(t) = \lim_{n\to\infty} [1 + (t/n)]^n. \tag{11.12}$$

This is the shape of the curve such that "N increases at a rate equal to N, that is, $dN/dt = N$".

This is an important function! Let us learn more about it.

First suppose you do your calculation of $N(t)$ in two parts. Suppose $t = t_1 + t_2$. You calculate

$$N(t_1) = N(0) \lim_{n\to\infty} [1 + (t_1/n)]^n = N(0)f(t_1).$$

Then you note this number down and take a break. Hit the pause button on your computer. Freeze your bacteria so they don't divide. Go and have a cup of tea. You come back and start the process again. Now your number of things to start with is $N(t_1)$. Let the process run for t_2 more seconds, until total running time $t = t_1 + t_2$. So we have:

$N(t) = N(t_1 + t_2) = N(t_1)f(t_2) = N(0)f(t_1)f(t_2)$

But as for any time t,

$N(t) = N(t_1 + t_2) = N(0)f(t_1 + t_2)$.

So our function, the shape of our curve, has a very simple property:

$$f(t_1 + t_2) = f(t_1)f(t_2). \tag{11.13}$$

This means that the function $f(t) =$"*(some number) to the power t*". But the curve we constructed and the function f are unique, there is only one such function $f(t)$. *So that "some number" is a unique particular number.* It is such an important number in mathematics, that, like the number π, it is given a special name. *our number is called e.* The function $f(t)$ is called "*the exponential function*", denoted $\exp(t)$, and given by

$$f(t) \equiv \exp(t) = e^t, \tag{11.14}$$

which is just "*e to the power t*". The number e itself can be calculated from the definition of the exponential function, since

$$e = e^1 = \exp(1) = \lim_{n\to\infty} [1 + (1/n)]^n. \tag{11.15}$$

Calculate *e*: try $n = 2, 3, 4, 5$, then try $n = 10, 100, 200...$ Make a table of your choice of n against the estimate of e for that n. It converges pretty rapidly.

So we know the solution of the equation $dN/dt = N$:

$$N(t) = N(0)\ \exp(t) = N(0)\ e^t. \tag{11.16}$$

From this it is simple to get the solution of the equation $dN/dt = C\,N$ with proportionality constant C. To do this we write:

$$\begin{aligned} dN/dt &= CN \\ dN/(Cdt) &= N \\ dN/dt' &= N \quad \text{defining} \;\; t' \equiv C\,t, \quad dt' \equiv C\,dt; \\ \text{so } N &= N(0)\ \exp(t') = N(0)\ \exp(Ct). \end{aligned} \tag{11.17}$$

So now we know how to get the solution for damping as in Equation 11.3, for population growth as described in Equation 11.4 and for chemical or nuclear decay, as in Equation 11.5. We leave the detailed solutions and applications to the problems. Returning to the case of damping, the solution is

$$\mathbf{v}(t) = \mathbf{v}(0) \exp(-t/\tau) \tag{11.18}$$

Velocity decreases because of the damping force. The smaller the velocity gets, the less the rate of decrease is. The velocity *decreases asymptotically* to zero. The position of the block is the sum (integral!) of all the little distances $dx(t) = v(t)\ dt$ it goes in short intervals of time dt, from t to $t + dt$. So, starting from $x(0) = 0$, and integrating, we find the position

$$\mathbf{x}(t) = \int_0^t \mathbf{v}(t')dt' = \mathbf{v}(0)\ \tau\ (1 - \exp(-t/\tau)). \tag{11.19}$$

Question: Why is the distance increasing only asymptotically ?

Until Galileo and Newton, it was thought that just to move an object, even at constant velocity, a force was required. Without any force the natural state of a body would be rest ($\mathbf{v} = 0$)!

Question: What led people to hold this opinion ?

CHAPTER 11 - PROBLEMS:

1. A population of mice, numbering $N(0) = 10$ initially, is increasing according to the law $dN/dt = N(t)$ where t is measured in months. Find and plot the population, approximately, at n intermediate times up to 4 months using Equation 11.10,
   (a) with $n = 4$ time intervals.
   (b) with $n = 8$ time intervals.
   (c) with $n = 16$ time intervals.
2. Plot the curves $f(t) = \exp(t),\ \exp(3.2t),\ \exp(-t)$ and $\exp(-t/5)$, versus t, using sample values obtained with a pocket calculator.
3. Can you get the curves in Problem 1 from the curve $f(t) = \exp(t)$ by changing units of t and/or changing the direction of your t axis?
4. An object of mass 10 kg is moving against a damping force $\mathbf{F_f} = -5\mathbf{v}$ force. The initial velocity is $\mathbf{v}(0) = 20\ \mathbf{i}\ m/s$.
   (a) Find the expression for $\mathbf{v}(t)$.
   (b) What is the velocity at $t = 2\ s, 6\ s, 10\ s$?
   (c) What is the acceleration at $t = 2\ s, 6\ s, 10\ s$?
   (d) Initial the object was at $\mathbf{x}(0) = 10\ \mathbf{i}\ m$. Find $\mathbf{x}(t)$.

Chapter 12

The Harmonic Oscillator

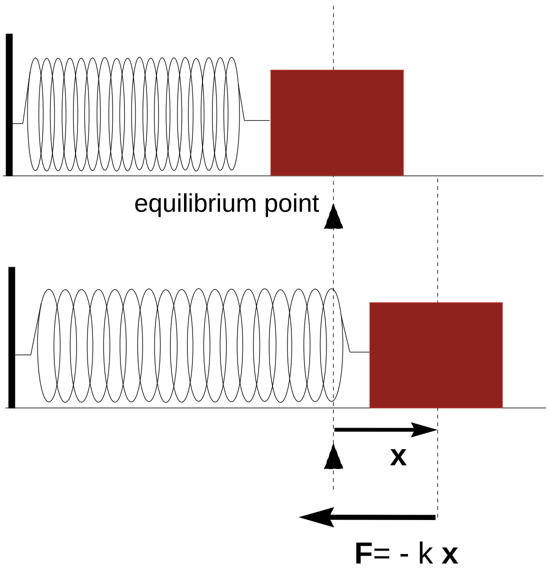


Figure 12.1:

A *perfect harmonic* spring applies a force,

$$\mathbf{F} = -k\mathbf{x}$$

on the mass when it is at displacement $\mathbf{x}$ from its equilibrium position as seen in the figure 12.1.

This force formula is called Hooke's Law. It is not a general law of nature, it is a property of springs and many other systems *when close to stable equilibrium.*

The equilibrium position, as we studied when we looked at potential energy curves, is where the mass will stay still if placed at rest initially.

In Hooke's Law the force is directed opposite to the displacement: it is a *restoring force.* When taken away from equilibrium the mass will be accelerated back toward the equilibrium position. The equilibrium point of the spring is a stable equilibrium point.

The restoring force of the spring has a very simple form: its magnitude is just proportional to $\mathbf{x}$: the restoring force is "linear in $\mathbf{x}$". This kind of force, $\mathbf{F} = -k\mathbf{x}$, is associated with the potential energy:

$$U = \frac{1}{2}kx^2$$

$$F = -\frac{dU}{dx} = -\frac{d}{dx}\left(\frac{1}{2}kx^2\right) = -kx$$

We study the spring in detail because it represents a situation that is common and important in nature and in technology. *Linear* restoring forces apply to the dynamics of any system *close to stable equilibrium.* In motions near the equilibrium point, systems move in and out through equilibrium under linear restoring forces.

Consider any potential $U(x)$ near a stable equilibrium point at $x = 0$ [1]. $U(x)$ has, in general, some particular shape with hills and valleys. Near the bottom of a valley, close enough to the stable equilibrium point (for small x), any potential $U(x)$ can be approximated by the potential $kx^2/2$. To see this, first look at a potential near a minimum. You can visualize that a curve like x^2 (which is called a *parabola*) has the right shape, a bowl shape, that can be fitted into the bottom of the valley by adjusting the constant $k/2$. At some small distance x from the position of the bottom, the potential energy U is approximately equal to the value of U at $x = 0$, right at the bottom, plus some corrections proportional to powers of x,

$$U(x) = U(0) + Ax + Bx^2 + Cx^3 + ... \tag{12.1}$$

You will learn in your Calculus courses that any function can be expressed with an expansion like Eq. (12.1) around its value at $x = 0$. We can chose the value of the potential energy at $x = 0$ to be 0: $U(0) = 0$. The expression for the force is:

$$F(x) = -\frac{dU}{dx} = -A - 2Bx - 3Cx^2 + ... \tag{12.2}$$

Now, $F(x = 0) = 0$; the force, the derivative of potential energy, is zero right at the equilibrium point. This shows that the constant $A = 0$. The approximate form of the potential energy near the equilibrium point is therefore:

$$U(x) = Bx^2 + Cx^3 + ...$$

Thus, for example, take $x = 0.1$.

$$U(0.1) \cong 0.01B + 0.001C$$

Unless C is very large compared to B, the last term is small and negligible. If C is greater than 10 B, the third term is not negligible at $x = 0.1$. But no matter how large C is, at even smaller x, at small *enough* x, the Bx^2 term will be dominant. Thus for small enough displacements near the equilibrium point

$$U(x) \cong Bx^2$$

[1] For example, go back and study the simulation in NS 101 SUCourse on potential energy.

With this definition the force is:

$$F(x) = -\frac{dU}{dx} = -2Bx \equiv -kx. \tag{12.3}$$

This is just the linear restoring force, Hooke's Law. The names of the constants do not matter. We identify $2B = k$, the conventional notation for the force constant in Hooke's Law.

Well, U can be any potential energy function with a stable equilibrium point! So near enough to the bottom of a *potential well*, near enough to the stable equilibrium point, any potential is approximately $(1/2)kx^2$; any potential gives a linear restoring force $F(x) = -kx$. The potential energy might be a function of many coordinates, of many particles. The variable of interest that affects the potential energy might be, for example, the angle between two chemical bonds, like the two OH bonds in the water molecule, and not even a distance like x. The principle is the same. In equilibrium the water molecule has a characteristic shape, with the two OH bonds making an angle of 104.5 degrees. The molecule oscillates about this equilibrium shape. For small deviations of the angle from 104.5 degrees, there is a restoring force back towards equilibrium. The magnitude of the force is just proportional to the small deviation of the angle from 104.5 degrees.

OK, $F = -kx$ is not just for the spring, it is general and important. *What kind of motion does it produce?*

$$F = ma = m\frac{d^2x}{dt^2} = -kx$$

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

so the solution $x(t)$ must be something whose second derivative with respect to time is a negative constant $(-k/m)$ times that same function $x(t)$!

We know already (from our study of uniform circular motion) two such functions, $\cos(\omega t)$ and $\sin(\omega t)$:

$$\frac{d^2}{dt^2}\cos(\omega t) = -\omega^2\cos(\omega t)$$

$$\frac{d^2}{dt^2}\sin(\omega t) = -\omega^2\sin(\omega t)$$

So a cosine or sine oscillation with angular frequency ω such that $\omega^2 = (k/m)$ is a possible motion of the spring/harmonic oscillator.

There is a simple and important property of linear systems called *the principle of superposition*. If you know two solutions like $x_1(t)$ and $x_2(t)$ then their linear combinations $Ax_1(t) + Bx_2(t)$ are also solutions. This works only if the equation is *linear*, which means there is just one power of x and one power of any derivatives of x in the equation. Newton's Second Law $F = ma$, which is the general law of mechanics, is *linear* on the ma side: it contains d^2x/dt^2 but not something like $(d^2x/dt^2)^3$ for example. On the left hand side (F) the equation is not linear in general; it is linear in the present very special and important case $F = -kx$. If the force is, for example, $F = -kx^5$, then the superposition principle does not work.

Now, our solution for the oscillator is

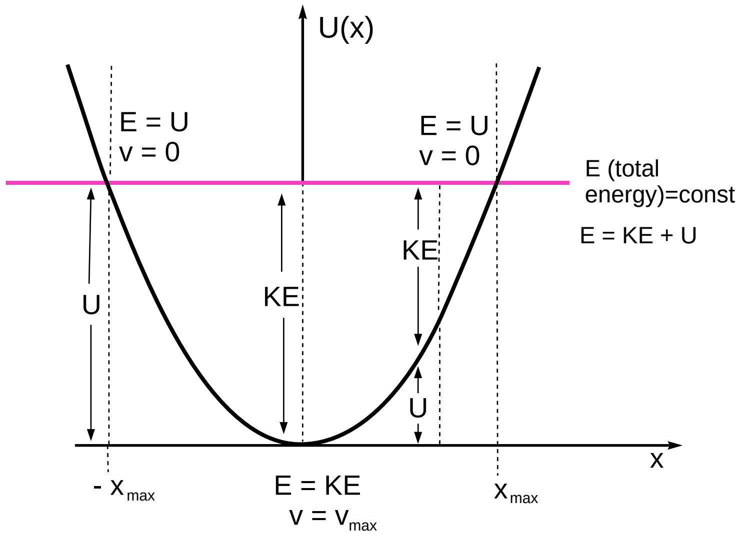


Figure 12.2:

$$x(t) = A\cos(\omega t) + B\sin(\omega t)$$

Is this the whole story? Or maybe there is another function, other than sine and cosine, that is also a solution and that you can add on because of the principle of superposition?

There is no solution other than sine and cosine. $F = ma$ allows you to determine the acceleration, which is the *second* derivative of x, at any moment. You can determine the full motion starting from an initial time $t = 0$ provided you know what $x(0)$ and $v(0)$ were. From these two constants (*the initial conditions*) you can determine the initial acceleration. Then that allows you to determine the velocity and the position after the interval dt, which gives you the new acceleration, which gives the next velocity and position and so on. So, knowing $x(0)$ and $v(0)$, these *two* constants (*"the initial conditions"*) determine the motion uniquely for ever after. The general solution $x(t) = A\cos(\omega t) + B\sin(\omega t)$ has two constants, A and B, multiplying the two independent functions sine and cosine. The two initial conditions $x(0)$ and $v(0)$ will tell you what A and B are. You cannot get a third constant, the coefficient of a third possible solution, from these two initial conditions. Therefore the two solutions are the full story.

The general solution can be written also in different forms, but always with two constants

$$x(t) = A\cos(\omega t) + Bsin(\omega t) = R\cos(\omega t + \phi) = R'\sin(\omega t + \phi')$$

(Do Problem 8 to see how this works.)

Let us now look at the mechanical energy during simple harmonic motion. Consider a block attached to a spring as in Figure 12.1. The potential energy of the block is seen in Figure 12.2. As the spring is stretched to x_{max}, potential energy $kx_{max}^2/2$ is stored in the spring. The total energy of the block is equal to this potential energy. In Figure 12.2, total energy is shown with the horizontal line[2]. When the block is released, it is accelerated

[2]This figure is from the "Harmonic Oscillator" simulation in NS 101 SUCourse. In the simulation you can

towards the equilibrium point and the potential energy is transformed to kinetic energy. If there is no damping mechanism in the system, the total mechanical energy that is potential energy + kinetic energy remains constant during motion. At $x = 0$, when the block is passing through the equilibrium point, $U = 0$ and the block's kinetic energy is maximum (see Problem 11).

CHAPTER 12 - PROBLEMS:

1. *Anharmonic Spring:* The potential energy of a mass at the end of a spring is given approximately as
$$U(x) = \frac{1}{2}kx^2 + Cx^3$$
where $U(x)$ is in units of Joules and x is the distance of the mass from the equilibrium position, in units of meters. The constants are $k = 200\ N/m$ and $C = 1000\ N/m^2$.
   (a) For what value of x are the two terms equal?
   (b) Will this spring obey Hooke's Law at $x < 0.01\ m$?
   (c) Plot the potential energy and the force, on both sides of the equilibrium point at $x = 0$ (ie. for both positive and negative x).
   (d) What is the force, and in what direction, when $x = 0.2\ m$? When $x = -0.2\ m$?
2. *Another Anharmonic Spring:* The potential energy of a mass at the end of a spring is given approximately as
$$U(x) = \frac{1}{2}kx^2 + Dx^4$$
where $U(x)$ is in units of Joules and x is the distance of the mass from the equilibrium position, in units of meters. The constants are $k = 200\ N/m$ and $D = 10000\ N/m^3$.
   (a) For what value of x are the two terms equal?
   (b) Will this spring obey Hooke's Law at $x = 0.01\ m$?
   (c) Plot the potential energy and the force, on both sides of the equilibrium point at $x = 0$ (ie. for both positive and negative x).
   (d) What is the force, and in what direction, when $x = 0.2\ m$? When $x = -0.2\ m$?
3. The position of a harmonic oscillator is given by $x(t) = 5cos(\pi t + \pi/5)$.
   (a) Find the position x where the speed is maximum ($v_{\max}$).
   (b) Find the positions x where the magnitude of the acceleration $a_{\max}$ is maximum.
4. Show, by substituting, that
$$x(t) = A\cos(\omega t) + B\sin(\omega t)$$

turn on damping as well. When there is damping, the total mechanical energy (potential + kinetic) does NOT remain constant.

is a solution of the equation of motion

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

provided that

$$\omega^2 = \frac{k}{m}$$

.

5. Suppose $x_1(t)$ and $x_2(t)$ are solutions of

$$m\frac{d^2x}{dt^2} = -kx^3$$

Is $Ax_1(t) + Bx_2(t)$ also a solution?

6. Consider a harmonic spring of mass m and spring constant k. The most general motion is given by

$$x(t) = A\ \cos(\omega t) + B\ \sin(\omega t)$$

where $\omega^2 = k/m$.

(a) ωt is called the *"Phase"* of the oscillation. At what values of the phase do the cosine and sine functions, and therefore the position $x(t)$ have the same value as at phase ωt_0?

(b) The time P after which the oscillator (the oscillating mass) returns to the same point, such that $x(t_0 + P) = x(t_0)$, is called the *period*. What is the period in terms of ω ?

(c) If the oscillator completes one cycle of oscillation in 0.05 seconds, how many cycles are done in 1 second? What is the value of ω?

(d) If the oscillator completes one cycle of oscillation in P seconds, how many cycles are done in 1 second? This is called the frequency f. Its unit is *cycles per second* or *Hertz, Hz*. What is the relation between the frequency f and the angular frequency ω ? The latter is called *angular frequency* or *radian frequency* and has units of *radians / second*.

7. Express A, B in Problem 5 in terms of the initial values $x(0)$ and $v(0)$.

8. The general harmonic oscillator motion $x(t)$ can be expressed also as

$$x(t) = A\ \cos(\omega t) + B\ \sin(\omega t) = R\ \cos(\omega t + \phi) = R'\ \sin(\omega t + \phi')$$

Express R, ϕ in terms of A, B . Express A, B in terms of R, ϕ. Do the same between R', ϕ' and A, B.

9. Plot $x(t) = \cos(\omega t)$, $x(t) = \sin(\omega t)$, $x(t) = \cos(\omega t + \phi)$ and $x(t) = \sin(\omega t + \phi')$ for arbitrary values of ϕ and ϕ' that you choose, other than 0 and $\pi/2$. These are all the same curve, same shape, only displaced in time t. These curves satisfy different conditions for $x(0)$ and $v(0)$.

10. From $x(t)$ in any representation you can get $v(t) = dx/dt$. Do this for each of the curves $x(t)$ in Problem 7. Now plot the corresponding curves $v(t)$.

11. In this problem you will calculate the kinetic energy and the potential energy of an oscillator with a mass of 1 kg on a spring, with spring constant k= 4 N/m. The motion is given by

$$x(t) = 2\sin(\omega t + \pi/6)$$

energy.

  (a) First find the value of ω.

  (b) What is the period of oscillation?

  (c) Calculate and plot the kinetic energy as a function of time t.

  (d) Calculate and plot the potential energy as a function of time t, on the same graph as the kinetic energy.

  (e) Calculate and plot the total energy. You will find that the total energy remains constant!

  (f) What is the average value of the potential energy? Of the kinetic energy? Can you see that both time average values are half of the total energy?

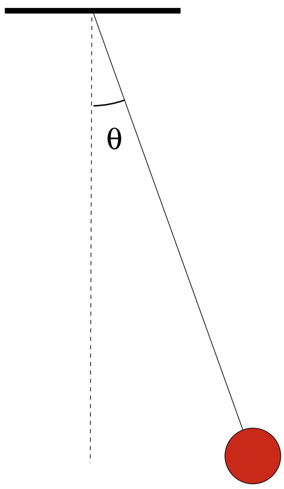


Figure 12.3:

12. A pendulum is making small oscillations with

$$\theta(t) = \theta_0 \cos(\omega t + \pi/2)$$

where $\omega^2 = g/L$, and $\theta_0 = 0.1$ rad. L, the length of the pendulum, is 10 cm. Take $g = 10\ m/s^2$.

  (a) Find the angular frequency ω and the period P.

  (b) At what times is the displacement $\theta = 0$?

(c) At what times is the displacement θ maximum, positive (towards the right in Figure 12.3)?

(d) At what times is the displacement θ maximum, negative (towards the left in Figure 12.3)?

(e) Find the expression for $\Omega(t) = d\theta/dt$, the angular velocity of the pendulum (not to be confused with the angular frequency ω).

(f) What is $\Omega(t)$ at each of the times in (b), (c) and (d), at the times when θ is zero, maximum negative, or maximum positive?

Chapter 13

Angular Momentum

Consider a body moving under the influence of another body which exerts a force $\mathbf{F}$. The position vector $\mathbf{r}$ is measured from the center of mass of these two interacting bodies – in other words the center of mass is chosen as the origin of the coordinate system. The equation of motion, Newton's 2nd Law, is:

$$\mathbf{F} = \frac{d(m\mathbf{v})}{dt} = \frac{d\mathbf{p}}{dt}. \tag{13.1}$$

An important conserved quantity, *angular momentum*, and the conditions when angular momentum is conserved can be derived from Newton's 2nd Law. To do this, let us take the cross product of both sides of Newton's 2nd Law with $\mathbf{r}$:

$$\mathbf{r} \times \mathbf{F} = \mathbf{r} \times \frac{d\mathbf{p}}{dt}. \tag{13.2}$$

Now remember the rule for taking the derivative of a product, which applies also for cross products:

$$\frac{d}{dt}(\mathbf{r} \times \mathbf{p}) = \mathbf{r} \times \frac{d\mathbf{p}}{dt} + \frac{d\mathbf{r}}{dt} \times \mathbf{p} \tag{13.3}$$

So the right hand side of Equation 13.2 can be written as

$$\mathbf{r} \times \frac{d\mathbf{p}}{dt} = \frac{d}{dt}(\mathbf{r} \times \mathbf{p}) - \frac{d\mathbf{r}}{dt} \times \mathbf{p}. \tag{13.4}$$

But $d\mathbf{r}/dt = \mathbf{v}$ and $\mathbf{p} = m\mathbf{v}$ are parallel vectors so their cross product vanishes, $(d\mathbf{r}/dt) \times \mathbf{p} = 0$. Thus we obtain:

$$\frac{d}{dt}(\mathbf{r} \times \mathbf{p}) = \mathbf{r} \times \mathbf{F}. \tag{13.5}$$

The vector $\mathbf{r} \times \mathbf{p}$, usually denoted with $\mathbf{L}$, is called the "*angular momentum*", and $\mathbf{r} \times \mathbf{F}$, usually denoted with $\mathbf{N}$, is called the "*torque*". Remember the definition of cross product: L contains the component of the momentum *perpendicular* to $\mathbf{r}$, so it is associated with "sideways", angular or rotational motion. The torque is likewise associated with the component of the force that is perpendicular to $\mathbf{r}$.

For some basic forces of nature, the force between two bodies is directed along the line joining them. This is true for the gravitational force as well as for attractive or repulsive electrostatic (Coulomb) forces, which we will learn about later. Such forces are called "*central*" forces. You may wonder why this is special: are there any forces that are *not* central? One important example is the magnetic force, which we will also learn about later. Forces that involve objects rotating in fluids like air or water can be non-central: the force between a tennis or pingpong racket and the ball, when you give the ball a spin, is also non-central. Leaving these exceptions aside, let us now return to central forces. Since the center of mass of two bodies is on the line joining them, a central force $\mathbf{F}$ is directed parallel to $\mathbf{r}$, the position of one of the bodies from the center of mass. Thus for central forces, the torque $\mathbf{r} \times \mathbf{F}$ is zero and we have

$$\frac{d\mathbf{L}}{dt} = 0. \tag{13.6}$$

Angular momentum is conserved for central forces.

What does this mean? The angular momentum $\mathbf{L}$ is a vector. When a vector is conserved *both its direction and its magnitude remain constant.* $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is perpendicular to both $\mathbf{r}$ and $\mathbf{p}$, so the direction of $\mathbf{L}$ is perpendicular to the plane formed by $\mathbf{r}(t)$ and $\mathbf{p}(t)$, sticking out of the plane in the direction found by the right hand rule that is used in the definition of the cross product: If you curl the fingers of your right hand from the direction of $\mathbf{r}$ towards the direction of $\mathbf{p}$, the direction of your right thumb indicates which way $\mathbf{L}$ will point. Since direction of the angular momentum $\mathbf{L}$ does not change, this means the plane containing $\mathbf{r}(t)$ and $\mathbf{p}(t)$ does not change with time. In a motion governed by central forces, so that the angular momentum $\mathbf{L}$ is constant, $\mathbf{r}(t)$ and $\mathbf{p}(\mathbf{t})$ must always remain in the same plane. So, for example, the orbit of a planet, which interacts with the Sun through the (central) gravitational force must always remain in the same plane. A planet does not move out of its plane (as seen in Figure 13.1, for example) because angular momentum is conserved.

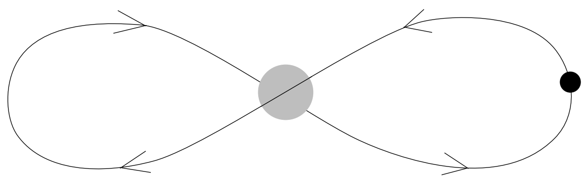


Figure 13.1:

The magnitude of $\mathbf{L}$ is:

$$L = rp_\perp = mrv_\perp \tag{13.7}$$

where $\perp$ denotes the direction perpendicular to $\mathbf{r}$. From Figure 13.2 we see that the component $v_\perp$ of the velocity that is perpendicular to $\mathbf{r}$ is

$$v_\perp = \frac{(dr_\perp)}{dt} = r\frac{d\theta}{dt} \tag{13.8}$$

where $d\theta/dt$ is the *instantaneous angular velocity.* Remember θ is in radians, so $dr_\perp = rd\theta$.

Thus we have

$$L = mr^2\frac{d\theta}{dt}. \tag{13.9}$$

Now, the area that the position vector sweeps per unit time is:

$$\frac{dA}{dt} = \frac{1}{2}r\frac{dr_\perp}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} \tag{13.10}$$

as can be seen from the Figure 13.2: In time dt, the position vector moves by an angle $d\theta$,

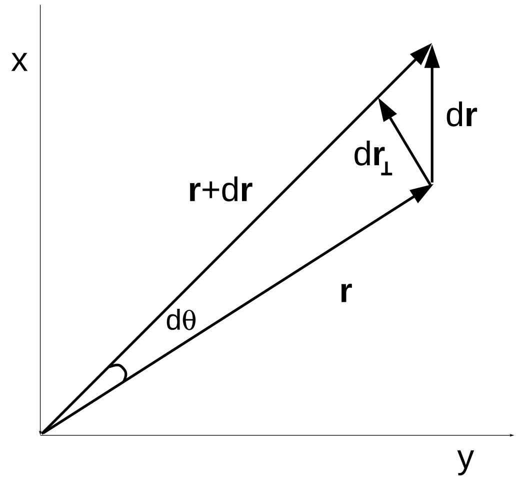


Figure 13.2:

sweeping the area $dA = r^2 d\theta/2$ of the triangle with base length r and height $dr_\perp = rd\theta$. Thus we have

$$L = 2m\frac{dA}{dt} \tag{13.11}$$

This is a general geometrical statement, whether L is conserved or not. When L is constant, this means $dA/dt = L/2m$ is also a constant:

The rate at which the moving body sweeps area in its orbit is constant in time.

Kepler's 2nd Law, which we will study in the Chapter on Gravitation and Kepler, states that

Equal areas are swept in equal times. (see Chapter 14)

Thus Kepler's 2nd Law is simply the observation that angular momentum is conserved in planetary motion, as it must be since the force of gravitation is a central force. Kepler arrived at his laws empirically, from analyzing Tycho Brahe's astronomical observations of the motions of the planets. The 2nd Law of Kepler follows from Newton's 2nd Law and the central nature of the gravitational force: it is an example of the conservation of angular momentum.

13.1 Symmetries and Conservation Laws - Noether's Theorem

This conservation law holds only if the total force is zero everywhere. But that means the potential energy is uniform [1] *everywhere*:

$$F_{tot} = -\frac{dU_{tot}}{dx} = 0 \;\; everywhere \qquad \rightarrow \qquad U_{tot} = constant \;\; (uniform\ in\ space)$$

That would mean there are no special points, no points where the potential energy is minimum or maximum, nor are there any places where the potential energy is a bit less or more from its value anywhere else. All points in space are equivalent. This is called a symmetry;

[1] "constant" is used for "not changing in time; "uniform" is used for "not dependent on position".

> "Symmetry under displacement": You go here or there and nowhere do you find anything different regarding the potential energy.

Momentum is conserved when there is symmetry under all displacements.

This is one example of a deep principle, called "Noether's [2] Theorem":

> If a system has some symmetry(ies) then the laws of dynamics yield conserved quantities, there will be things that do not change with time, or "constants of motion."

A Conservation Law will apply if there is a symmetry.

CHAPTER 13 - PROBLEMS:

1. What is the angular momentum of a particle of mass m in uniform circular motion with period P on a circle of radius r?
2. What is the rate at which the area under the radius vector increases in circular motion with radius r and angular velocity ω ?
3. Kepler discovered that planets move in elliptic orbits with the Sun at a focus of the ellipse.
   (a) According to the conservation of angular momentum, is a planet moving fastest or slowest when it is closest to the Sun?
   (b) What is the ratio of the speed v_1 of the planet at r_{min} to its speed v_2 at r_{max}?
   (c) What is the ratio of the angular velocity $\omega = d\theta/dt$ at r_{min} to that at r_{max}?
4. When you are riding a bicycle, the spinning bicycle wheels stay straight due to the conservation of angular momentum.
   (a) When the bicycle is running, which way does the angular momentum of the spinning wheel point?
   (b) If you lean to the left while riding, which way would you exert "torque" ($\mathbf{r} \times \mathbf{F}$)?
   (c) The torque causes the change in angular momentum of the wheel (see Equation 13.5). Which way is the change in angular momentum?
   (d) What is the direction of the final angular momentum?
   (e) How does this effect the direction of the wheel?

[2]Emmy Noether was a 19th century mathematician. She proved the general relation between symmetries and conservation laws for Newton's laws of dynamics. The Noether Theorem also applies in quantum mechanics and in relativity

Chapter 14

Gravitation and the Kepler Problem

Newton discovered the detailed form of one of the fundamental forces of nature, *Gravitation*, which is exerted by any mass on any other mass. Gravitation is a rather weak force, but it has a long range. Gravity is important between large masses. It is the force that determines the motions of the Earth, the Sun, the Moon, the planets and the stars – *"the Heavenly Bodies"*. The force of gravity between two masses is proportional to both masses and to the inverse square of the distance between the two masses:

$$\mathbf{F}_{G;2,1} = -\frac{GM_1M_2}{r^2}\ \hat{\mathbf{e}}_{r;2,1} \tag{14.1}$$

In this equation $\mathbf{F}_{G;2,1}$ is the gravitational force that the body of mass M_2 exerts on the body of mass M_1, and G is Newton's Gravitational Constant. Its value in SI units is $G \cong 6.67 \times 10^{-11} m^3 s^{-2} kg^{-1}$.

The same mass appears both as *gravitational mass* in the force of gravity, and as *inertial mass* in the "ma" part of Newton's Second Law. The mass therefore cancels out from both sides of Newton's Second Law in gravitational problems. *All masses fall with the same gravitational acceleration, as realized first by Galileo.*

Kepler's Laws about the motion of the planets follow from *Newton's Second Law*, which is the general *equation of motion* under any force, and *Newton's Law of Gravitation.* In fact Newton arrived at his Law of Gravitation working backward, from Kepler's Laws. The full derivation of Newton's Law of gravitation from Kepler's Laws for elliptic orbits is simple in principle, but a bit complicated in the full mathematical details. The background and the full derivation are given in the NS 101 extra lecture, "Yıldızlar, İnsanlar ve Matematik" by Tosun Terzioğlu[1]. Here we will study the Kepler problem for a planet in a circular orbit.

Kepler arrived at his laws of planetary motion empirically, by studying the data collected by Tycho Brahe[2]. **Kepler's Laws state:**

[1] available on NS101 SUCourse. See also Terzioğlu, T. ,"Gökten Bir Elma Düştü", Matematik Dünyası,2006-1, pg 67-71.

[2] See Gribbin, Chapter 3 for the fascinating story of Kepler.

> I. Planets move around the Sun in elliptical orbits with the Sun at one focus of the ellipse.
>
> II. The line connecting a planet to the Sun sweeps equal areas in equal times as the planet moves in its orbit around the Sun.
>
> III. The square of the period[3] of a planetary orbit is proportional to the cube of the semi-major axis of the orbit.

Closed orbits of objects affected by the force of gravity are ellipses, including the case of the special ellipse, the *circle* with eccentricity $e = 0$, and very eccentric ellipses $e \lesssim 1$, like the orbits of *comets*. Orbits under gravity can also be *open orbits*. Open orbits are *hyperbolae* of various eccentricities e greater than 1 or the parabola, $e = 1$, in the special case of an object moving with the *escape velocity*. All these orbits are curves called *conic sections*.[4]

For a circular orbit, Kepler's First Law becomes just the statement that the Sun is at the center of the circular orbit. Since the distance from the Sun is constant, just equal to the radius, for circular motion, Kepler's Second Law means the planet moves along the circle with constant angular speed. This is uniform circular motion.

Kepler's Third Law then follows by using Newton's Second Law. The force is the gravitational pull of the Sun on the Earth. For uniform circular motion, the acceleration is

$$\mathbf{a}(t) = -\omega^2 \mathbf{r}(t) = -\omega^2 r \hat{\mathbf{e}}_r(t) \tag{14.2}$$

where $\mathbf{r}$ is the position vector directed from the Sun to the Earth. The (uniform) angular speed ω of the Earth is related to the Earth's orbital period $P = 1$ year around the Sun, $\omega = 2\pi/P$.

Using this special form of the acceleration for uniform circular motion together with the gravitational force, Equation 14.1, Newton's Second Law becomes

$$\frac{GM_{Sun}}{r^2} = \omega^2 r = \frac{4\pi^2 r}{P^2} \tag{14.3}$$

This can be expressed as

$$P^2 = \frac{4\pi^2}{GM_{sun}} r^3 \tag{14.4}$$

This does not depend on the mass of the Earth, only on the mass of the Sun. The Earth's mass has dropped out of the equation because of the very important property of gravity, the equality of gravitational mass to inertial mass. The proportionality constant for Equation 14.4 depends only on the mass of the Sun. Equation 14.4 therefore has the same form for any planet in circular orbit – it is nothing but Kepler's Third Law for circular orbits. For a planet in elliptical orbit it turns out that Equation 14.4 still holds with only the replacement of the circle's radius r with half the long axis (the semi-major axis, a),

$$P^2 = \frac{4\pi^2}{GM_{sun}} a^3 \tag{14.5}$$

[3] the period of the orbit means the time it takes the planet to complete one tour of the orbit. The period is the *year* for that planet.

[4] See "Matematiğin Aydınlık Dünyası", Sinan Sertöz, TÜBİTAK Popüler Bilim Kitapları.

CHAPTER 14 - PROBLEMS:

1. The Earth completes its orbit around the Sun in one year. The Earth's orbit is an ellipse of very small eccentricity, almost a circle. Half the length of the major axis, *semi-major axis*, is approximately equal to the average distance of the Earth from the Sun, called an *astronomical unit, a.u.*, 1 a.u. $\cong 1.5 \times 10^8$ km . Look up the length of the year for Jupiter. What is the size of Jupiter's orbit: express the semi-major axis of Jupiter's orbit in a.u. and in km.

2. **Newton, Apple and Moon**: Newton realized that an apple falls from the tree towards the center of the Earth, due to the gravitational force. He also realized that the gravity force must reach the Moon, and explained the Moon's orbital motion as a constant fall towards the Earth, just as a cannonball fired horizontally with enough speed would keep falling towards the Earth without reaching the ground.

   By comparing the accelerations and the distances of the apple and of the Moon to the center of the Earth, he realized that the gravity force must obey an "inverse square law". Let us look at this: Assume that the Moon is in a circular orbit around the Earth, with a radius of 3.8×10^8 m, and a period of 28 days.

   The radius of the Earth is 6400 Km.

   $g = 9.8 m/s^2$.

   (a) Calculate the acceleration with which the Moon "falls toward" the Earth.

   (b) What is the ratio of the Moon's acceleration to that of the apple?

   (c) What should this ratio be according to Newton's Universal Law of gravitation?

3. (* – Difficult – Z. Gedik)

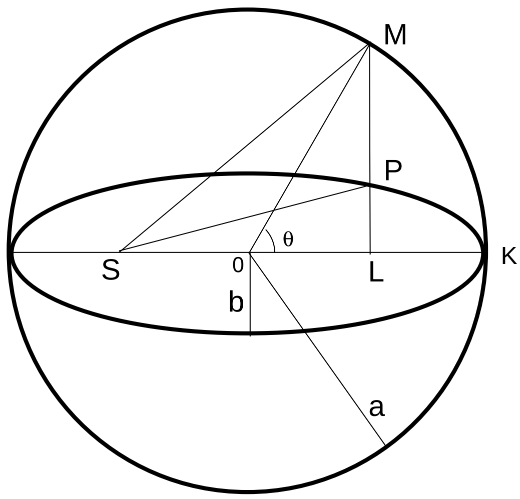


Figure 14.1:

According to Kepler's first law the orbit of each planet P is an ellipse with the Sun S at one focus. The second law states that the area $A(t)$ of the slice (SKP) changes uniformly so that , where a and b are semi-major and semi-minor axes of the ellipse, respectively, and T is the period. Point P of the ellipse can be obtained from a circle of radius a, as shown in Figure 14.1, by dividing each line segment [ML] such that

$$\frac{|\mathrm{LP}|}{|\mathrm{LM}|} = \frac{b}{a}$$

and that is why the area of the ellipse is given by $ab\pi$ which is nothing but the area of the circle πa^2 multiplied by the ratio b/a. It can be easily shown that $|\mathrm{S0}| = \sqrt{a^2 - b^2}$.

(a) Explain why $A(t) = \dfrac{b}{a}\, A(\mathrm{SKM})$ where $A(\mathrm{SKM})$ is the area of the slice SKM.

(b) Using $A(\mathrm{SKM}) = A(\mathrm{SOM}) + A(\mathrm{OKM})$ show that

$$A(\mathrm{SKM}) = \frac{a^2}{2}\left(\sqrt{1 - \frac{b^2}{a^2}}\sin\theta + \theta\right)$$

(c) After verifying

$$\frac{2\pi}{T}t = \sqrt{1 - \frac{b^2}{a^2}}\sin\theta + \theta$$

evaluate $\dfrac{2\pi}{T}t$ for $\theta = 0, \pi/2, \pi$ and interpret your result.

Exam Problems - Mechanics

1. **[Fall 2008, Midterm]** A child of mass M, is on a merry-go-round, moving in a circle, with also an up and down motion, so that her position is

$$\mathbf{r} = R\cos(\omega t)\mathbf{i} + R\sin(\omega t)\mathbf{j} + h\sin(5\omega t)\mathbf{k}$$

Given that $\frac{d}{du}(\cos u) = -\sin u$ and $\frac{d}{du}(\sin u) = \cos u$,

(a) Find the velocity vector $\mathbf{v}(t)$.

(b) Find the acceleration vector $\mathbf{a}(t)$.

(c) Find the total force $\mathbf{F}(t)$ acting on the child.

(d) The small displacement $d\mathbf{r}$ traveled during a short time interval dt is $d\mathbf{r} = \mathbf{v}(t)dt$. Calculate the work done, $\mathbf{F} \cdot d\mathbf{r}$.

2. **[Fall 2008, Midterm]** A ball of mass M hanging on a string with tension force $\mathbf{T}$, is moving with uniform speed in a circular track of radius R (see Figure 14.2).

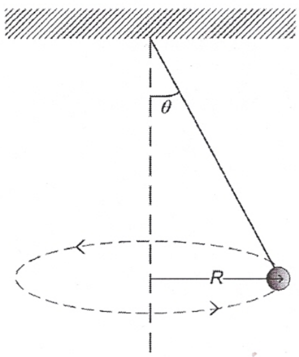


Figure 14.2:

(a) Show all the force vectors on the diagram.

(b) Write the horizontal and vertical components of the equation of motion $\mathbf{F} = m\mathbf{a}$.

(c) What is the speed v in terms of θ, R and the gravitational acceleration g?

(d) What is the angular speed ω?

(e) What is the period P?

3. **[Fall 2008, Midterm]** A block is moving with initial speed v on the frictionless track shown in Figure 14.3.

(a) Mark the stable equilibrium points on the figure.

(b) Mark the unstable equilibrium points.

(c) What is the minimum value of v_i if the block is to reach point B?

(d) At what point is the kinetic energy largest?

(e) What is the final speed v_f at the point F?

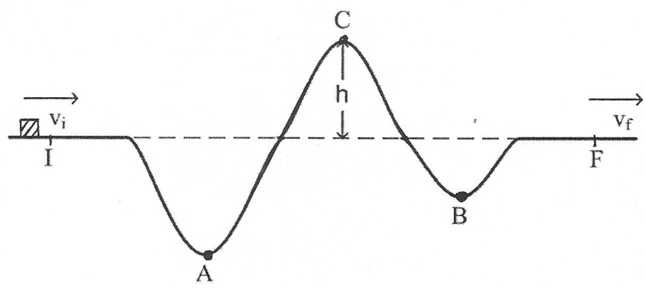


Figure 14.3:

4. **[Fall 2008, Midterm]** A block of mass 3 kg is initially released from rest at the top of an inclined plane of length $18\sqrt{2}\ m$ and inclination angle $\theta = 45°$ ($\sin\theta = 1/\sqrt{2}, \cos\theta = 1/\sqrt{2}$), as shown in Figure 14.4.
The magnitude of the friction force is given by $F_f = 0.1F_{\mathrm{N}}$, where F_{N} is the normal force exerted on the block by the plane. Take the gravitational acceleration $g = 10\ ms^{-2}$.

   (a) What is the magnitude of the normal force?

   (b) What is the work done by the normal force during this motion?

   (c) What is the work done by friction during this motion?

   (d) What is the work done by gravity during this motion?

   (e) What are the initial and final values of the gravitational potential energy, U_i and U_f?

   (f) What is the final speed of the block?

5. **[Fall 2008, Midterm]** The potential energy $U(r)$ between two oxygen atoms can be approximated by

$$U(r) = \frac{1}{r^2} - \frac{2}{r}$$

where $U(r)$ is measured in electron-volts (eV, 1 eV $= 1.6 \times 10^{-19}$ Joule) and the distance r between the atoms is measured in Angstroms ($1\,\rho A = 10^{-10}\ m$).

   (a) How does $U(r)$ behave at large r (as r goes to infinity)?
   At very small r (as r goes to zero)?
   At what point is the potential energy $U(r)$ zero?
   Sketch the potential energy $U(r)$.

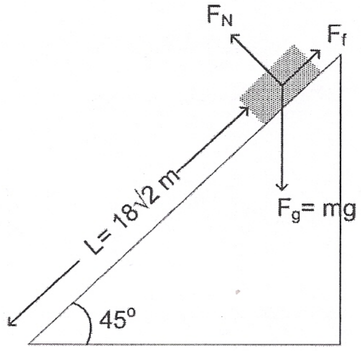


Figure 14.4:

(b) Find the force $F(r)$ between atoms as a function of distance r. At what point is the force $F(r)$ zero?

(c) Find the minimum value of the energy, $U_{\min}$. At which position, r_0, does the energy become minimum?

6. **[Fall 2007, Midterm]** Suppose that a point particle moving in three dimensions has the position vector $\mathbf{r} = (2t^2 + 1)\,\mathbf{i} - t^3\,\mathbf{j} + 5t\,\mathbf{k}$ in m/s.

   (a) Find the position of the particle at $t = 0$ and $t = 2\ s$.

   (b) What is the displacement of the particle between $t = 0$ and $t = 2\ s$?

   (c) Find the velocity of the particle as a function of time.

   (d) What is the magnitude of the velocity at $t = 1\ s$?

   (e) Find the acceleration vector as a function of time.

7. **[Fall 2007, Midterm]** An object of mass $M = 1\ kg$ is sliding on a track as shown in Figure 14.5. The object is initially at rest at an height $h = 5\ m$ as shown in the figure. The surface of the track is frictionless except between the points A and B, where the kinetic friction coefficient is $\mu = 0.2$ between the surfaces of the track and the object. (Take $g = 10\ m/s$)

   (a) Find the speed of the object at point A.

   (b) What is the friction force when the object it is between the points A and B?

   (c) Find the work done by the friction force between the points A and B.

   (d) What is the kinetic energy at point B ?

   (e) The potential energy of a spring compressed by a distance x is $U(x) = \frac{1}{2}kx^2$. What must be the value of the spring constant k for the spring shown in the figure to stop the mass M with a compression of $0.4\ m$.

8. **[Fall 2007, Midterm]** Two blocks of masses $M_1 = 2\ kg$ and $M_2 = 10\ kg$ are connected with a rope. The mass M_2 is being pulled by a force F parallel to the inclined plane as shown in Figure 14.6. The pulley shown in the figure is frictionless, is fixed and has

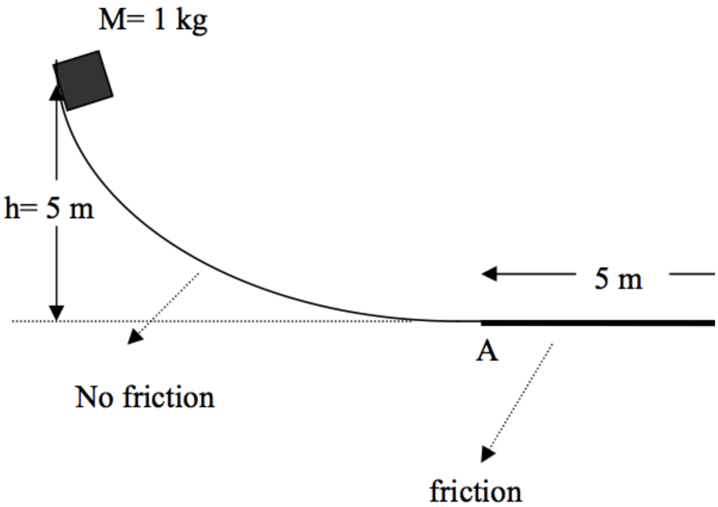


Figure 14.5:

negligible (zero) mass. For (a) to (d), assume that there is no friction between the blocks and the surfaces of the planes. ($\sin 37° = \cos 53° = 3/5$ and $\cos 37° = \sin 53° = 4/5$)

(a) Draw the free-body diagrams for M_1 and M_2.

(b) Write down the equations of motions for M_1 and M_2.

(c) Find the common magnitude of the acceleration of the masses.

(d) Find the magnitude of the tension force T on the rope.

(e) Now, suppose that there is friction between M_1 and the surface of the horizontal plane (the inclined plane is still frictionless). What is the minimum friction coefficient μ required for the masses to remain at rest?

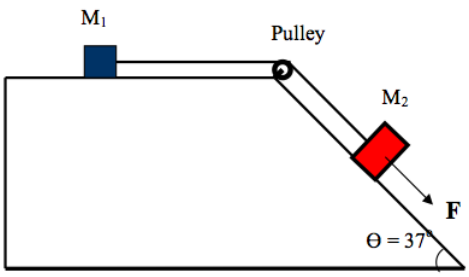


Figure 14.6:

9. **[Fall 2007, Midterm]** Suppose that two planets A and B, with masses m_a and m_b , are moving around a star of mass M in circular orbits with radii R_A and R_B respectively. The gravitational force of interaction of the planets with each other is negligible. The mass of the star M is much greater than the masses of the planets, so that the center of the star can be taken as the center of the circular orbits of both planets.

(a) Write down the equation of motion, $F = ma$, for each of the planets, writing the forces and accelerations.

In terms of the orbital radii R_A and R_B of the planets, find the ratio of the planets'

(b) orbital velocities V_A and V_B.

(c) orbital periods P_A and P_B.

(d) angular velocities (ω_A and ω_B).

(e) kinetic energies (K_A and K_B).

(f) potential energies (U_A and U_B).

10. **[Fall 2007, Midterm]** An object of mass $m = 1\ kg$ is sliding on a frictionless ice surface with an initial speed $v_i = 1\ m/s$ as shown in Figure 14.7. The ice surface ends with a frictionless circular profile. The radius of the circular part is $1\ m$. At a critical angle, $\Theta = \Theta_{\text{crit}}$ the mass m will slide off from the surface of ice. Take $g = 10\ m/s$.

    (a) Draw the free body diagram for the block when it is on the circular surface at an angle Θ from the vertical, and write the equation of motion $F_{\text{total}} = ma$ for the circular motion.

    (b) Find the critical angle Θ_{crit}.

    (c) What is the speed of the block at $\Theta = \Theta_{\text{crit}}$?

Figure 14.7:

11. **[Fall 2005, Midterm]** Two objects of mass m are connected by a string going over a frictionless pulley as shown in Figure 14.8. There is a constant frictional force F between the surface of the inclined plane and the mass on the right. The masses are released from rest. The string remains fully stretched while the mass on the right travels a distance d up the inclined plane, at which point the masses are moving with speed v.

    (a) What is the initial total mechanical energy of the system?

    (b) What is the final total mechanical energy of the system?

    (c) Write down the work-energy theorem for this problem in terms of the known quantities and the frictional force F. What is the sign of the work done by the frictional force?

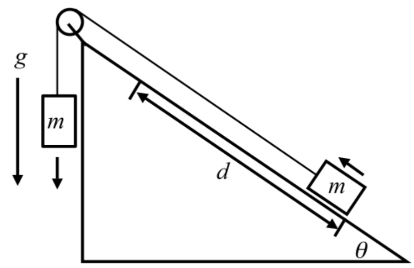


Figure 14.8:

(d) Express F in terms of m, g, v and θ.

12. **[Fall 2005, Midterm]** A mass m is released from rest at position angle θ_0 from the vertical in a frictionless semispherical bowl, as shown in Figure 14.9.

(a) What is the initial energy?

(b) What is the kinetic energy when the mass is at angle θ?

(c) Consider small oscillations of the mass near the bottom where $\theta \ll 1$ radian, so that $v \approxeq v_x$, $x = R\sin\theta \approxeq R\theta$ and $\cos\theta \approxeq 1 - \frac{1}{2}\theta^2$. Show that the total mechanical energy can be written in the form

$$E \approxeq \frac{1}{2}mv_x^2 + \frac{1}{2}kx^2.$$

(d) Remember that the angular frequency $\omega = \sqrt{\frac{k}{m}}$ for the spring-block system. Plug in the k you found in part (c) and thus obtain ω for this system.

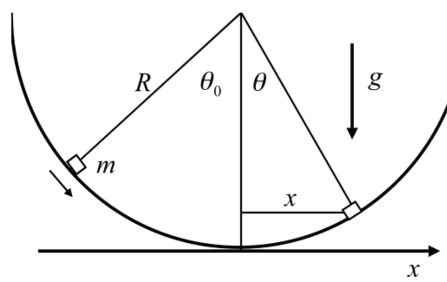


Figure 14.9:

13. **[Fall 2005, Midterm]** For a planet of mass m moving in the gravitational field of a large star of mass M, the total mechanical energy E is given by

$$E = \frac{1}{2}mv_\|^2 + \frac{l^2}{2mr^2} - \frac{GMm}{r}$$

where $v_\| = v_r$ is the velocity component in the radial direction, r is the distance between the planet and the star, $l = mv_\perp r$ is the angular momentum, and G is the gravitational constant. Here $v_\perp = v_\theta$ is the component of the velocity perpendicular to the radial direction.

(a) Assuming that $l \neq 0$ sketch the effective radial potential

$$V(r) = \frac{l^2}{2mr^2} - \frac{GMm}{r}$$

(b) Find the radius $r_{\min}$ for which $V(r)$ attains its minimum value by equating the derivative of $V(r)$ to zero.

(c) Show that $r_{\min}$ is nothing but the radius of the circular orbit where the centripetal acceleration $a = v_\perp^2/r$ is provided by the gravitational force $F_G = GMm/r^2$.

(d) Next, find the radius r_0 for which $V(r)$ vanishes.

(e) Using your result from part (d), verify that $v_\perp$ is nothing but the escape velocity

$$v_{\text{escape}} = \sqrt{\frac{2GM}{r_0}}$$

if $E = 0$. What kind of orbit does the mass m now follow?

14. **[Fall 2004, Midterm]** A $m = 10\ g$ bullet traveling at $v = 1010\ m/s$ strikes a $M = 1\ kg$ block which is initially at rest, pushing it backwards (see Figure 14.10).

(a) What is the velocity of the block (plus bullet) just after it is hit by the bullet?

(b) The block hits against a spring attached to a wall and comes to a stop. If the spring has a force constant k given by $1.01 \times 10^4\ N/m$, how much is the spring compressed when the block stops? What is the potential energy stored in the spring?

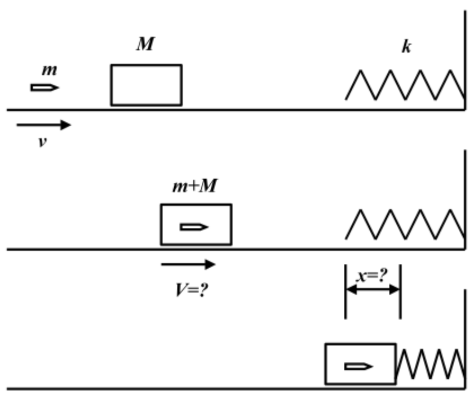


Figure 14.10:

15. **[Fall 2004, Midterm]** A $1\ kg$ mass attached to a spring on a frictionless horizontal table is pulled out from equilibrium by $5\,cm$ and released from rest. The mass then oscillates with a period $T = 1\,s$.

(a) Find the angular frequency ω and the spring constant k.

(b) Find the position $x(t)$ and velocity $v(t)$ as functions of time.

(c) Find the kinetic energy as a function of time.

(d) Find the potential energy as a function of time.

(e) Find the total energy from (c) and (d).

16. **[Fall 2004, Midterm]** A particle of mass m is subjected to an attractive force $F = -Ar^3$ in the radial direction.

(a) Find the potential energy $U(r)$ such that $-dU/dr = F$.

(b) Is the angular momentum constant? Why? Does Kepler's Second Law (Equal areas are swept in equal time...) hold in this problem?

(c) Find the angular speed ω for a circular orbit of radius r under this force.

17. **[Fall 2009, Midterm]** A model rocket is fired vertically from rest with a constant total vertical acceleration of $4.0\,m/s^2$ continuing for $5.0\,s$. Its fuel is then finished, so it continues to go upward as a free particle and then falls back down. Take the acceleration due to the Earth's gravity to be $g = 10\,m/s^2$.

(a) What is the velocity and the altitude of the rocket at time $t = 5.0\,s$?

(b) What is the maximum altitude reached?

(c) What is the total time from takeoff until the rocket reaches the maximum altitude?

(d) What is the total time from takeoff until the rocket strikes the ground?

(e) What is the speed of the rocket when it strikes the ground?

18. **[Fall 2009, Midterm]** A block of mass M is moving at uniform speed v on a circular track of radius R on a frictionless surface (see Figure 14.11).

(a) Show all the force vectors on the diagram.

(b) Write the horizontal and vertical components of the equation of motion $\mathbf{F} = M\mathbf{a}$.

(c) What is the speed v in terms of θ, R and the gravitational acceleration g?

(d) What is the angular speed ω and the period P?

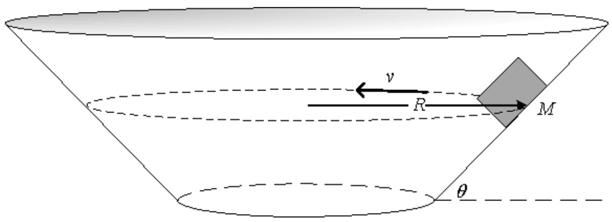


Figure 14.11:

19. **[Fall 2009, Midterm]** A $10\,kg$ body is moving along the x axis. The position changes with time as $x(t) = t^4$.

(a) Find the velocity $v(t)$.

(b) Find the acceleration $a(t)$.

(c) Find $F(t)$, the total force acting on this body as a function of time.

(d) Express the time t in terms of the position x of the body at that time, and substituting in $F(t)$, find the force as a function of position, $F(x)$.

(e) Find the potential energy $U(x)$.

(f) Find the total work done on the particle from $t = 0$, $x = 0$ to $t = 1$, $x = 1$.

20. **[Fall 2009, Midterm]** A block of mass $2\,kg$ is moving in a one dimensional track with springs such that the potential energy is $U(x) = 16x^2 - 2x^3 + x^4$, in $Joules$. There is no friction.

(a) What is the approximate form of the potential energy for "small" distances x from the equilibrium point $x = 0$? How small should x be for this approximation?

(b) What is the frequency ω of the oscillations about the equilibrium point?

(c) At time $t = 0$ the block is released from rest at $x = 2\,m$. What is the total energy of the block?

(d) What is the position $x(t)$ as a function of time?

(e) Now suppose some dust is placed on the track so that there is a constant friction force of $F_f = 8\,Newtons$. The block is again released from rest at $x = 2\,m$. What is the total distance the block will travel back and forth before it stops at the equilibrium point $x = 0$?

21. **[Fall 2009, Midterm]** A satellite of mass m is in a circular orbit around the Earth. The speed of the satellite is v_0 and the radius of the orbit is r_0 . The mass of the Earth is M.

(a) Write the equation of motion, $\mathbf{F} = m\mathbf{a}$ for this circular motion.

(b) What is the gravitational potential energy $U(r)$?

(c) Use the equation of motion to show that the total energy $E_0 = \alpha U(r_0)$. What is the value of α?

(d) What is the angular momentum vector $\mathbf{L}$? Draw a figure and show the direction of $\mathbf{L}$.

(e) Express the kinetic energy and the total energy in terms of the angular momentum L, m, and r_0.

22. **[Spring 2010, Midterm]** A 1-kg object initially at rest moves from $\mathbf{r_1} = (3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k})\ m$ to $\mathbf{r_2} = (5\mathbf{i} - 6\mathbf{j} + 10\mathbf{k})\ m$ under the influence of $\mathbf{F} = (\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})\ N$ in 2 seconds.

(a) Find the average velocity, $\mathbf{v}$.

(b) Find the acceleration $\mathbf{a}$.

(c) Find the velocity at $t = 2$ s.

(d) Find the total work by the force done on the object.

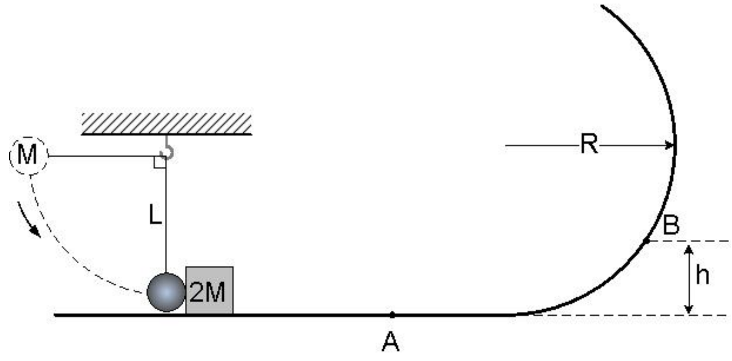


Figure 14.12:

23. **[Spring 2010, Midterm]** A ball of mass M suspended with a rope of length L is raised and released as shown in Figure 14.12. The ball collides with a block of mass 2M at rest elastically and sets the block into motion on a frictionless surface.

    (a) Find the potential energy gain of the ball before it is released.

    (b) Find the velocity of the block at point A.

    (c) Find the maximum height that the block can reach.

    (d) Find the centripetal acceleration on the block as it passes the point B, whose height above the horizontal (h) is a quarter of L (h = L/4).

24. **[Spring 2010, Midterm]** A system of 3 masses as shown in Figure 14.13 is pulled by a force, F. The horizontal surface is frictionless while there is friction between masses M_2 and M_3 (static friction coefficient: μ_s and kinetic friction coefficient: μ_k).

    (a) Draw the free body diagram for each mass.

    (b) Calculate the acceleration of the mass, M_1 in case M_3 remains on M_2.

    (c) Determine the critical force, $F = F_c$ that would make the mass, M_3 move.

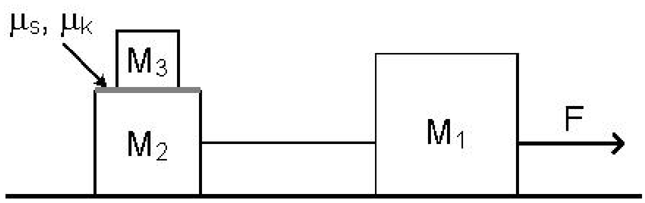


Figure 14.13:

25. **[Spring 2010, Midterm]**

    (a) A race car is driving around a circular race-track with a constant speed of 280 km/h. Explain briefly the physical force(s) acting on the car.

(b) A block is held on a frictionless incline by a massless rope, as shown in Figure 14.14. Draw the free body diagram for the block.

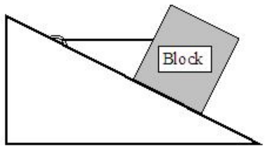


Figure 14.14:

(c) A block slides down on a frictionless ramp as shown in Figure 14.15. How would the acceleration change in time? Explain briefly the reason of your answer.

Figure 14.15:

(d) A compact car and a large truck collide head on and stick together. Which one undergoes the larger momentum change?

(e) A ball elastically bounces off the floor as shown in Figure 14.16. What is the direction of the momentum exchange, $\Delta\mathbf{p}$? Explain why.

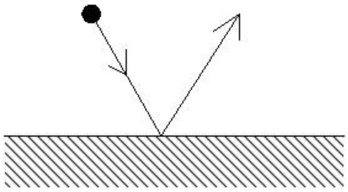


Figure 14.16:

26. **[Fall 2010, Midterm 1]** A point mass m is moving on a XY plane, and the position is given as a function of time as follows:

$$\mathbf{r}(t) = 3t\sin(\omega t)\mathbf{i} \; + 3t\cos(\omega t)\mathbf{j}$$

(a) Find the velocity $\mathbf{v}(t)$ as a function of time.

(b) Find the acceleration $\mathbf{a}(t)$ as a function of time.

(c) Find the force $\mathbf{F}(t)$ acting on the mass as a function of time.

(d) Find the work done W on the mass m from $t = 0$ to 1s.

27. **[Fall 2010, Midterm 1]** A ball with mass $m = 1\,\text{kg}$ is thrown with $45°$ angle to the horizontal plane with a velocity of $v_0 = 10\text{m/s}$. Assume $g = 10\text{m/s}^2$.

    (a) Find the maximum height the ball can reach.

    (b) Find the maximum distance the ball travel before it hits the ground.

    (c) Find the ball's velocity when it hits the ground. (Hint: Velocity is a vector!)

    (d) Find the work done on the mass during this motion: from start until it hits the ground.

28. **[Fall 2010, Midterm 1]** Suppose that we live in a different universe, where the gravitational force between two masses m_1 and m_2 separated by r is given by:

$$F(r) = -\frac{G' m_1 m_2}{r^4}$$

, where G' is a new gravitational constant.

    (a) Find the gravitational potential energy $U(r)$ in this universe.

    (b) Sketch this potential energy $U(r)$ as a function of r.

    (c) Find the escape velocity, v_e, of an object on the Earth, which has the mass M_E and radius R_E.

29. **[Fall 2010, Midterm 1]** Small balls with mass m and $5m$ are suspended on two strings, as shown in Figure 14.17. The mass with $5m$ is lifted up by height h and is released, and it goes down and hits the other mass. The collision is completely inelastic: they stick together at the collision.

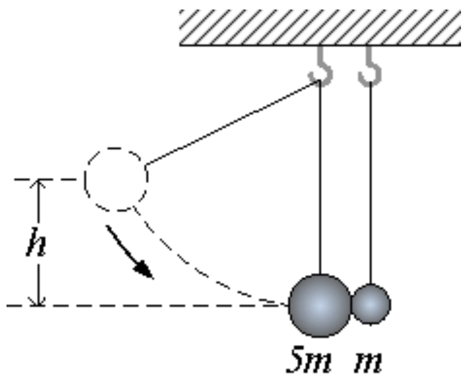


Figure 14.17:

    (a) Find the velocity of the mass, $5m$, just before the collision.

    (b) Find the velocity of the combined mass, $6m$, just after the collision.

(c) Find the maximum height the combined mass reaches.

(d) Calculate the amount of energy lost during the collision.

30. **[Fall 2010, Midterm 1]** A child of mass M_c is sliding from a frictionless slide while she holds a pendulum of mass m_p in her hand as shown in Figure 14.18. The slide makes $30°$ angle to the horizontal plane. Assume $g = 10\,\mathrm{m/s^2}$. [$\sin 30° = 1/2$ and $\cos 30° = \sqrt{3}/2$]

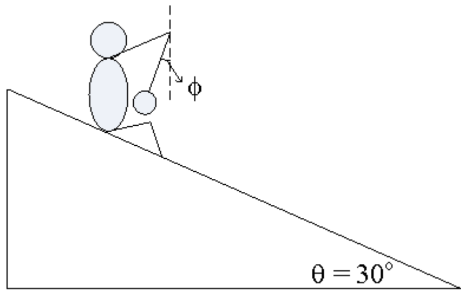


Figure 14.18:

(a) Draw the free body diagram of the child and pendulum.

(b) Find the acceleration of the child-pendulum system.

(c) Find the angle (ϕ) that pendulum makes with the vertical axis.

31. **[Fall 2010, Final]** A block of mass M rests on a frictionless table and is connected to a horizontal spring of spring constant k, as shown in Figure 14.19. The other end of the spring is connected to the wall. A second small block of mass m sits on the larger mass M. The coefficient of static friction between the blocks is μ_s. The blocks are released from x = A, and oscillate.

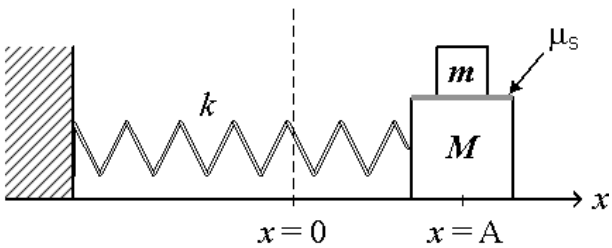


Figure 14.19:

(a) Draw a free-body diagram for each mass when the blocks are at x = A.

(b) Find the maximum amplitude of the oscillation such that the top block m does not slide on the larger block M.

32. **[Spring 2011, Midterm 1]** A small rocket of mass 100 kg is moving horizontally with a velocity of 500 m/s in the x direction. The total external force on the rocket is zero.

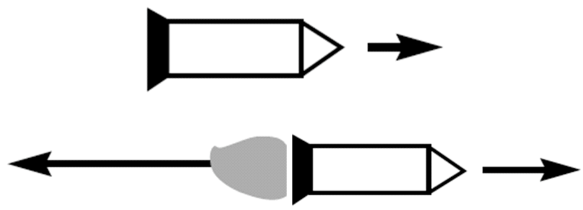


Figure 14.20:

10 kg of jet gas is ejected at a velocity of $-400\,\mathrm{m/s}$ in the x direction (see Figure 14.20). The ejection takes place at a constant rate over a time interval $\Delta t = 10\,\mathrm{s}$.

(a) What is the total momentum of the rocket before the jet is ejected?

(b) What is the total momentum of the rocket + jet system after the jet is ejected?

(c) What is the velocity (magnitude and direction) of the rocket after the jet is ejected?

(d) What is the magnitude and direction of the force that the jet exerts on the rocket during the ejection?

33. **[Spring 2011, Midterm 1]** Two blocks, 1 and 2, of the same mass M are released from rest at the same moment $t = 0$ from the top of two frictionless inclined planes of the same height $h = 0.2\,\mathrm{m}$. The inclination angles of the two planes are $\theta_1 = 45°$ and $\theta_2 = 30°$. ($\sin 45° = \sqrt{2}/2$; $\sin 30° = 1/2$. Take $g = 10\,\mathrm{ms}^{-2}$).

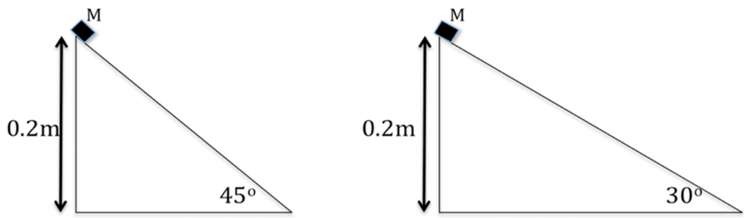


Figure 14.21:

(a) What is the speed of block 1 when it reaches the bottom of its track?

(b) What is the speed of block 2 when it reaches the bottom of its track?

(c) What is the acceleration of block 1 down the incline?

(d) What is the acceleration of block 2 down the incline?

(e) Which block reaches the bottom first?

(f) A block of mass m is released from rest at the top of an inclined plane of height h, inclination angle θ and coefficient of kinetic friction, μ. Find the time it takes to reach the bottom in terms of g, h, θ and μ.

34. **[Spring 2011, Midterm 1]** A sled (kızak) follows the track shown in Figure 14.22.

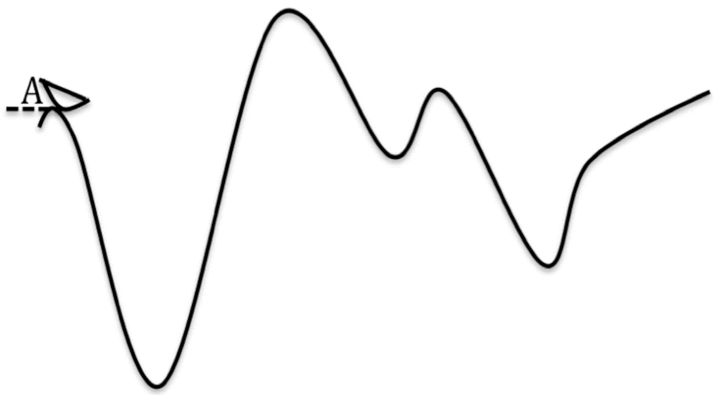


Figure 14.22:

(a) Show the equilibrium points on the track.

(b) Are any of these stable equilibrium points? Which one(s)?

(c) The sled is released from rest at point A. Indicate on the track the farthest point that it can reach, assuming that the track is frictionless.

(d) At what point does the sled have maximum kinetic energy?

(e) Now suppose there is some friction on the track. Describe the motion. Can the sled reach as far as it did when there was no friction? Where will it stop?

35. **[Spring 2011, Midterm 1]** A toy car of mass $M_1 = 1\,\text{kg}$ is kept moving on a circle of radius 1 m by the tension force of magnitude T applied by a string tied to a hanging mass $M_2 = 0.4\,\text{kg}$, which remains at rest (see Figure 14.23). The toy car is pushed by a toy engine which applies a force of magnitude $F_e = 20$ Newtons in the direction tangential to the circle. This is opposed by a friction force $F_f = -\alpha v$, where v is the velocity vector which is tangential to the circle. (*In this problem F_f is NOT related to the normal force*). Take the gravitational acceleration $g = 10\,\text{m/s}^2$.

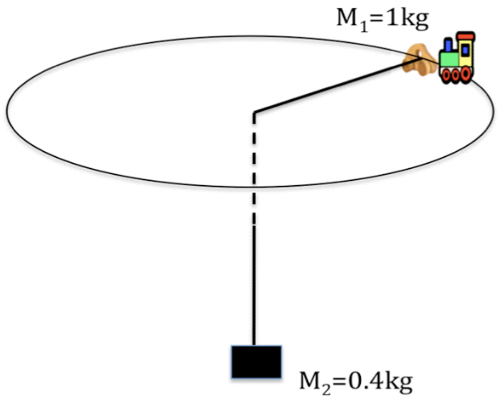


Figure 14.23:

(a) Draw the free body diagram for the hanging mass.

(b) Find the tension T.

(c) Draw the free body diagram for the toy car.

(d) What is the speed v of the toy car?

(e) What is the value of the constant α? What are its units?

36. **[Spring 2011, Midterm 1]** A mass of 8 kg is moving in a one dimensional track attached to an anharmonic spring with potential energy is $U(x) = 16x^2 - x^4$, in Joules. There is no friction.

    (a) For what values of x is $U(x) = 0$? For what values of x, is $U(x)$ maximum or minimum? Sketch the potential energy curve $U(x)$.

    (b) What is the approximate form of the potential energy for "small" distances x from the equilibrium point $x = 0$? How small should x be for this approximation?

    (c) What is the frequency ω of the small oscillations about the equilibrium point?

    (d) At time $t = 0$ the block is released from rest at $x = 2$ m. What is the total energy?

    (e) Some dust is placed on the track so that there is a constant friction force of magnitude $F_f = 8$ Newtons. The block is again released from rest at $x = 2$ m. What is the total distance the block will travel back and forth before it stops at the equilibrium point $x = 0$?

Chapter 15

Pressure – Volume – Temperature: The Ideal Gas

In this Chapter we will derive the relationship between pressure, volume and temperature of a *dilute* gas. In such a gas, atoms[1] collide with each other, and with the walls of the container. We call the container a "box", and show it as a cube, although the actual shape of the box is not important. The total volume occupied by the gas atoms in the box is very very small compared to the volume of the box. The average distance between atoms is much much larger than the size of the atoms. This is what we mean by a dilute gas.

We also assume that the potential energy of interaction between the atoms is negligible in comparison to their kinetic energies. No interatomic or intermolecular forces need to be taken into account. This is a very good assumption in a dilute gas because the basic interaction that binds electrons to atoms and keeps atoms together in molecules, the electrostatic Coulomb interaction is *screened*. The electron distribution in atoms and molecular bonds screens the bare Coulomb forces exerted by the positively charged nuclei. The total force exerted by a neutral atom or molecule on other atoms and molecules is very very very weak because of the screening; the cancelation of the effects of positive and negative charges. Moreover, the range of the effective very weak total force is very short, of the order of the size of the atoms or molecules. Unless another atom comes extremely close, as in a very rare, nearly head-on collision, it will not feel a force, there is effectively no interaction. Finally, we take the collisions between atoms or between atoms and the walls of the box to be *elastic*. This means that no energy is lost from the total kinetic energy during the collision: the very weak interaction during even the closest collisions means no energy is converted to the internal energy of the atoms to get the atoms into excited states. All these assumptions are valid for a dilute gas. So the dilute gas has only kinetic energy. Such a gas is called an *"ideal gas"*. The ideal gas model is a *simple* model, and more importantly it is a *realistic* model for many situations in nature and in technology, for gases inside stars or in engines, for example. A dense gas, where the interatomic distance is not much larger than the atomic size, is not an ideal gas. As density increases, such a gas is not microscopically very distinct from a liquid or an amorphous solid or a glass; these are all strongly interacting systems. The differences between a dense gas, a liquid and an amorphous solid relate to properties like viscosity.

Question: An ionized dilute gas (a dilute plasma) is not ideal. Why?

[1]– or molecules: we shall call the particles in the gas "atoms", to mean either atoms or molecules, except where the distinction between atoms and molecules is important.

We write Newton's Second Law as

$$\mathbf{f} = \frac{\Delta(m\mathbf{v})}{\Delta t} \tag{15.1}$$

where $\mathbf{f}$ is the average force acting on an atom during the time interval Δt, m is its mass, $\mathbf{v}$ its velocity and t is the time. Now consider only the x-component of the motion. Suppose an atom has a velocity v_x, to the right, and it hits the right wall of the cube, and bounces back with velocity $-v_x$, since its kinetic energy does not change during the collision. The change in momentum (*the impact*) is

$$\Delta(m\mathbf{v}) = 2mv_x \tag{15.2}$$

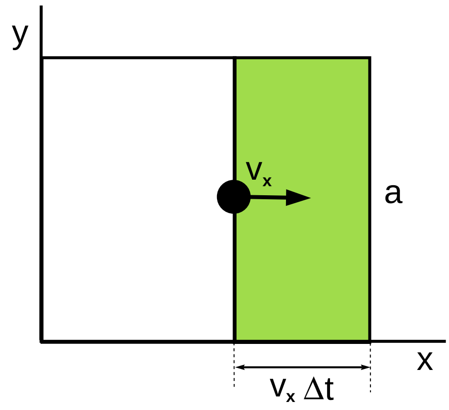


Figure 15.1:

Let us consider the x-y projection of the box shown in Figure 15.1. Each side of the box has length a. In this figure we see one gas molecule moving to the right with velocity v_x. The molecules will have different velocities. The magnitudes of their velocities, as well as their directions will be different. But for the moment let us assume that they all have the same speed v_x along the positive or negative x-axis. (We will take care of differences in velocities later). A molecule moving with this velocity will travel a distance $v_x\Delta t$ during the time interval Δt. Among all molecules which are in the shaded volume $a^2v_x\Delta t$ half are moving to the right. These molecules will hit the right face of the box during the time interval Δt. Let the total number of particles in the box be N. Since the molecules are randomly distributed in the box, there are

$$\frac{(1/2)a^2v_x\Delta t}{a^3}\ N\ = (1/2)a^2v_x\Delta t\frac{N}{V} \tag{15.3}$$

molecules which will hit the right wall within the time interval Δt. Here $V = a^3$ is the volume of the box. Each molecule exerts a force of

$$\mathbf{f} = \frac{\Delta(mv_x)}{\Delta t} = \frac{2mv_x}{\Delta t} \tag{15.4}$$

on the right wall. The total force is

$$F = \frac{1}{2}\left(\frac{2mv_x}{\Delta t}\right) a^2 v_x \Delta t \frac{N}{V}$$

$$F = a^2 m v_x^2 \frac{N}{V}\,. \tag{15.5}$$

The force per unit area, F/a^2 is just *Pressure, P*.

The SI unit of pressure, N/m^2 is given the name *Pascal, Pa* in honor of the 17th century French philosopher, mathematician and scientist Blaise Pascal. Another common unit of pressure is the *atmosphere, atm*, referring to the average pressure of the Earth's atmosphere at sea level. 1 atm = 1.013 × 10^5 Pa.

Therefore we have the relation

$$P = m v_x^2 \frac{N}{V} \tag{15.6}$$

This is a relation between the pressure P, the density $n \equiv N/V$ and something related to the kinetic energy of an atom. Now, not all molecules have the same velocity, and neither are they all moving in the same direction. The term v_x^2 refers to the average of the square of the x-velocity of the molecules.

Question: Is "the average of the velocity squared" the same thing as "the square of the average velocity"? What is the average x-velocity in the gas?

The total velocity squared is the sum of the squares of the x, y, and z-components of the velocity,

$$v^2 = v_x^2 + v_y^2 + v_z^2 \tag{15.7}$$

Since the velocity components of the many atoms are not all the same, we have to take averages. Taking the average is denoted by angular brackets $\langle\ \rangle$. The average values of the squared velocity components are related simply as

$$\left\langle v^2 \right\rangle = \left\langle v_x^2 \right\rangle + \left\langle v_y^2 \right\rangle + \left\langle v_z^2 \right\rangle \tag{15.8}$$

But the molecules do not know the difference between the x, y, and z-directions. There is no preferred direction of the motion. The average mean squared values $\left\langle v_x^2 \right\rangle$, $\left\langle v_y^2 \right\rangle$ and $\left\langle v_z^2 \right\rangle$ must be equal, so that

$$\left\langle v_x^2 \right\rangle = \left\langle v_y^2 \right\rangle = \left\langle v_z^2 \right\rangle = \frac{1}{3}\left\langle v^2 \right\rangle \tag{15.9}$$

Substituting equation (15.9) in the pressure equation, equation (15.6), we obtain

$$P = \frac{1}{3} m \left\langle v^2 \right\rangle \frac{N}{V}. \tag{15.10}$$

The average kinetic energy of the molecules is $\frac{1}{2} m \left\langle v^2 \right\rangle$. So the pressure can be written as

$$P = \frac{2}{3}\left(\frac{1}{2} m \left\langle v^2 \right\rangle\right) \frac{N}{V} \equiv \frac{2}{3}\frac{U}{V} \equiv \frac{2}{3}\ u \tag{15.11}$$

where we have introduced U, *the total "internal" energy of the gas*, and u, *the energy density*, which is the energy of all the molecules in one unit volume of the gas.

Let us now return to Equation 15.10. We can write Equation 15.10 as

$$PV = N(\frac{1}{3}m \langle v^2 \rangle) \tag{15.12}$$

This already looks like the **Ideal Gas Law**, the relation between P, V, N and *the temperature* T first determined empirically in the 17th Century by Boyle, Charles and others[2]:

$$\boxed{PV = NkT \qquad (15.13)}$$

except that our Equation 15.12 does not refer to the temperature T. Instead, it contains the average kinetic energy per atom. We already have in Equation 15.10 the proportionality between the pressure P and the density $n = N/V$ ("Charles' Law").

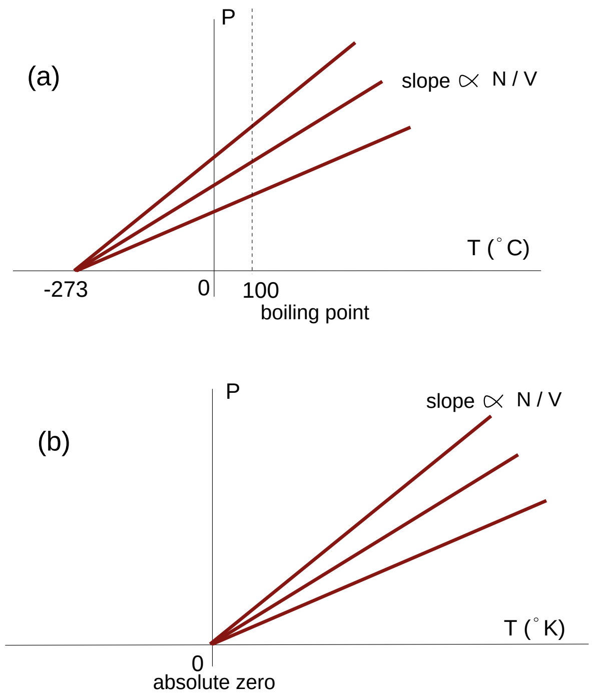


Figure 15.2:

Pressure P, volume V and the number of atoms N (or mass, or number of moles of gas) in the box are all positive quantities. So the temperature T must also be positive in the correct (*"absolute"*) scale. Indeed the empirically measured ideal gas relation between P and the temperature T, for a fixed amount of gas, i.e., constant $n = N/V$, is as shown in Fig 15.2 (a), with T given in degrees *Celsius*. Also called *Centigrade* ($^\circ C$), this temperature scale is obtained by defining the freezing point of water (under "standard" atmospheric pressure) to

[2]see for example, Gribbin Chapter 4

be 0 degrees and the boiling point to be 100 degrees. When a shift by 273 degrees is made, introducing the *Kelvin* ($^\circ K$) temperature scale;

$$T(^\circ K) = T(^\circ C) + 273, \tag{15.14}$$

the relation becomes the Ideal Gas Law in its familiar form, as shown in Fig 15.2 (b). *The Ideal Gas Law holds with temperature expressed in the absolute (Kelvin) scale.*

Now compare the Ideal Gas Law, Equation (15.13, with the relation we derived, Equation (15.12. The average kinetic energy of atoms in the gas cannot be negative. Its minimum value is zero, just like the absolute temperature. So the absolute temperature, as defined empirically, corresponds to the average kinetic energy of the atoms:

$$kT(^\circ K) = \frac{1}{3} m \left\langle v^2 \right\rangle. \tag{15.15}$$

The proportionality constant k, the Boltzmann constant, has the value $k \cong 1.38 \times 10^{-23}$ Joules $^\circ K^{-1}$. We have derived the Ideal Gas Law from basic mechanics, and understood that temperature, the empirical measure of what we experience as hotness and coldness is actually a measure of the energy per atom.

The number N of atoms or molecules can be expressed as $N = n_{mole} \times N_{Avogadro}$, where n_{mole} is the number of *moles* and the Avogadro number $N_{Avogadro} \cong 6.02 \times 10^{23}$ is the number of atoms (or molecules) in a mole. The Ideal Gas Law then has the form

$$PV = n_{mole} N_{Avogadro} kT \equiv n_{mole} RT \tag{15.16}$$

The *gas constant* $R = N_{Avogadro} k$ has the value 8.315 $J/(mole\text{-}^\circ K)$.

Note that pressure and temperature are emergent concepts in the sense we discussed in Chapter 3: they are defined in terms of average value of the kinetic energy in the macroscopic ideal gas. The derivation of the Ideal Gas Law is based on Newton's 2nd Law, as applied to the individual microscopic particles, but pressure, temperature, and density are properties of the ideal gas. Like the Ideal Gas Relation they satisfy, these concepts have no meaning for the microscopic particles.

CHAPTER 15 - PROBLEMS:

1. A box of volume 1 m^3 contains one mole of steam, which means the Avogadro number, $\cong 6.02 \times 10^{23}$ of water molecules.
   (a) What is the number density n, the number of molecules per m^3?
   (b) What is the average distance between neighboring molecules? Compare this with the typical size scale of atoms and small molecules, 10^{-10}m. Is this a dilute gas?
2. What is the source of the atmosphere's pressure? What is the total mass of the column of the Earth's atmosphere above 1 m^2 at sea level? Assume that the gravitational acceleration g at the Earth's surface is constant throughout the thickness of the atmosphere.
3. A box of volume 1 m^3 contains air at a temperature of 27 °C at a pressure of 1 atmosphere. Assume that 80% of the molecules in air are N_2 and 20% are O_2 molecules.

(a) Are the nitrogen and oxygen molecules at the same temperature? Why?

(b) How many molecules are in this box?

(c) What is the density of oxygen molecules?

(d) What is the density of nitrogen molecules?

(e) How much of the pressure is due to oxygen and how much is due to nitrogen? These are called *"partial pressures"*

4. (a) What is the average (*"root mean square"*) speed of the O_2 molecules at temperature 27 °C? What is the average speed of the N_2 in air at temperature 27 °C?

   (b) In an ideal gas containing different kinds of molecules, how does the average speed of each kind of molecule depend on the molecular mass?

5. A container with a movable piston contains an ideal gas at temperature 20°C in a volume of 1 liter at a pressure of 1.8×10^5 Pa.

   (a) Find the number of moles of the gas in the container.

   (b) The gas pushes against a piston and expands the gas to twice its original volume while its pressure drops to atmospheric pressure (1.0×10^5 Pa). What is the final temperature?

Exam Problems - Thermodynamics

1. **[Fall 2005, Midterm]** An industrial oven of volume V_{oven} is initially at $27^\circ\ C$. We wish to increase the temperature to $927^\circ\ C$. Since the oven is not airtight, the pressure inside is the same as the atmospheric pressure P_{atm}. It follows from $P_{atm}V_{oven} = NkT$ that the only way to increase the temperature T is to drive some of the molecules out of the oven.

   (a) Convert the initial and the final temperatures to Kelvins.

   (b) Find the number of the molecules to be driven out in terms of the initial number N.

   (c) Calculate the ratio
   $$\frac{v_{rms}(T = 927^\circ C)}{v_{rms}(T = 27^\circ C)}$$
   where v_{rms} denotes the root mean square velocity.

   (d) Does the total molecular internal energy of the gas *in the oven* increase, decrease or stay the same? You must explain your answer.

2. **[Fall 2004, Midterm]** Pressure P, volume V, temperature T and number of moles n of an ideal gas are related to each other by $PV = nRT$ where $R = 8.3$ J/mol-K is the universal gas constant. This can also be written as $PV = NkT$ where N is the number of atoms in the gas and $k = 1.4 \times 10^{-23}$ J/K is the Boltzmann constant. According to the atomic model of the ideal gas composed of N atoms of mass m,
   $$P = \frac{2}{3}\left(\frac{N}{V}\right)\left(\frac{1}{2}m\langle v^2\rangle\right)$$
   where $\langle v^2\rangle$ is the average of the square of the velocity of the atoms.

   (a) What is the relation between the temperature and the average kinetic energy per atom?

   (b) If the volume of the gas is reduced from V to $V/3$ at constant temperature, by pushing in a piston slowly, what happens to the energy of the gas (= "internal energy")?

   (c) The work done by the piston on the gas is given, for this process at constant temperature, by the formula:
   $$W = nRT\ln\left(\frac{V_i}{V_f}\right)$$

where V_i and V_f are the initial and final volumes. Evaluate the work done on 5 moles of ideal gas at temperature 200 K. Do not evaluate the logarithm numerically.

(d) According to the work-energy theorem, the work done should be converted to energy. Where does this energy go? Interpret your result using the first law of thermodynamics, $\Delta E = Q + W$, where ΔE is the change in internal energy of the gas, Q is the heat absorbed by the gas and W is the work done on the gas by the piston.

3. [**Spring 2010, Midterm**] One mole of ideal oxygen gas at a pressure $8.3 \times 10^5 N/m^2$ is in a container of volume 1 L.

   (a) Calculate the temperature of the gas.

   (b) Calculate the average kinetic energy of an oxygen atom.

   (c) Calculate the average speed of an oxygen atom.

   (Take $R = 8.3\ JK^{-1}mol^{-1}$; $k = 1.4 \times 10^{-23}\ JK^{-1}$; $N_{\text{Av}} = 6 \times 10^{23}$; 1 mole of oxygen is 16 grams)

4. [**Fall 2010, Midterm 2**] According to the equipartition of energy, the average kinetic energy of a molecule moving in one dimension is $\frac{1}{2}kT$, where k is the Boltzmann constant and T is the temperature. Molecules in a gas can move in three dimensions with an average speed in each dimension being almost equal to each other.

   (a) What is the average speed, v, of a molecule in the gas?

   (b) Show that, for an ideal gas at a temperature T,

   $$\frac{dv}{v} = \frac{1}{2}\frac{dT}{T},$$

   where v is the average speed of the molecules in three dimension.

   (c) Using the approximation, $\frac{dv}{dT} \approx \frac{\Delta v}{\Delta T}$, calculate the change in the average speed of air molecules (Δv) in terms of the initial speed v_i, if the air temperature changes from 30°C in winter to 24°C in summer.

   Does the average speed increase or decrease from winter to summer?

Chapter 16

Coulomb's Law

Experiments show that pieces of matter have a property, called *charge*, which is responsible for electric (and, as we see later, magnetic) phenomena. There are two kinds of charge, "positive" and "negative". These opposite kinds of charge attract each other, while objects with the same kind of charge repel each other with the "electrostatic" force. The electrostatic force is a long range force like gravity but it is much stronger than gravity. This force is responsible for binding matter, holding atoms, molecules, liquids and solids together usually in a neutral state, with zero total charge.

16.1 Fundamental Particles, the Quantum of Charge, and Our Units of Charge

Charge, like mass, is a property of fundamental particles, like the electron, the proton and the neutron, which make up nuclei (and atoms, and other fundamental particles which are unstable when isolated). Such fundamental particles can exist briefly in very high energy collisions produced artificially in accelerators. They are also produced naturally in the centers of stars like our Sun, in the vicinity of very energetic black holes, in the collisions of *cosmic rays* with matter in the Earth's atmosphere. The cosmic rays themselves are stable energetic particles or energetic *photons*. The fundamental particles that are unstable in vacuum may be stable in very dense or energetic conditions. All energetic fundamental particles were stable and abundant at very early times, the first seconds and minutes after the Big Bang[1]. The "familiar" neutron, which makes up about half, by mass, of ordinary matter, is unstable in vacuum, decaying into a proton, an electron and a massless particle called the "*neutrino*" after a lifetime of about 27 minutes. Neutrons are stable in nuclei where they actually serve to bind the nuclei. The repulsive electrostatic force between the protons in the nuclei is overcome by *the strong nuclear force*, which works best for about equal numbers of neutrons and protons in nuclei.

Of the three basic building blocks of nuclear and atomic matter, the neutron has zero charge, while the proton and the electron have equal but opposite charges, + or $-e$. Protons have positive charge $+e$ and electrons have negative charge $-e$. The signs of the two particles' charges are chosen by convention. The important experimental fact is that the proton and the electron have opposite charges.

It turns out, from experiments, that all other fundamental particles also have positive or negative (or zero) charges that are *only integer multiples* of the electron and proton

[1]see for example, "The First Three Minutes", S. Weinberg; Turkish translation: "İlk Üç Dakika", TÜBİTAK Popüler Bilim Kitapları

charge *quantum* e. Quarks, which are the particles that protons and neutrons are made of, have charges of $2/3\ e$ and $-1/3\ e$, but quarks are never isolated and it is implied by the experiments and present theories that they never can be. In any case, in "ordinary" matter conditions charge is quantized in multiples of e.

Atomic nuclei have positive charge, because they contain protons and neutrons but not electrons. A nucleus with Z protons has charge $+Ze$. It attracts and binds Z electrons, to make a neutral atom. Matter under ordinary conditions is neutral, it is made of neutral atoms. Some atoms are *ionized*[2] in solutions. In plasma, the highly ionized state of matter, usually at high temperatures, pressures or densities, there may be very few neutral atoms left. Still, even plasmas and solutions are neutral in the bulk, macroscopically, while microscopically a plasma or solution contains equal numbers of positively and negatively charged particles. In any macroscopic volume containing matter the total charge on positive ions, including isolated nuclei and protons, usually equals the total charge on negatively charged ions and electrons. This is because the Coulomb force, which is the strongest long range force, does not easily allow separation of negative charges from positive charges. A macroscopic piece of matter can be put into a *polarized* state, where its negative charges are concentrated to one side and the other parts are left with positive charge. Typical examples are provided by rubbing a piece of glass or plastic against silk or wool. This is an unstable state and is usually temporary, with the charges moving about to achieve neutrality everywhere, unless a special design is made to supply opposite charges to different parts of the system and to prevent the motion of charges towards neutralization.

The SI unit of charge, the Coulomb, abbreviated as C, is a typical measure of the amount of charge that can be separated to one side of a macroscopic system. The natural unit, the charge quantum e is

$$e = 1.6 \times 10^{-19}\ \mathrm{C}.$$

A separation of 1 C of charge actually involves moving about a very small fraction of the electrons in a macroscopic sample (see Problem 1).

16.2 The Electrostatic (or Coulomb) Force

In Figure 16.1 we see the force applied on a charge q by charge Q. The force vector on the charge q is indicated by an arrow[3].

By Newton's 3rd Law, the charge q applies an equal and opposite force on the charge Q, which we could denote by an arrow on Q, equal in length to the arrow shown on q but pointing in the opposite direction.

As you drag q nearer to the charge Q, the force becomes larger. It is always pointing away from the charge Q if both charges have the same sign, both positive or both negative (as in the top panels of Figure 16.1): *Like charges repel.*

If the charge q is negative while the charge Q is positive then the force on each charge points towards the other charge: *Opposite charges attract.* The bottom panels in Figure 16.1 show the forces for two opposite charges.

[2] an "ion" is an atom or molecule that has positive or negative charge. A positively charged ion is formed when an atom or molecule loses some of its electrons. When extra electrons are attached to an atom or molecule, this is a negatively charged ion.

[3] The Coulomb force simulation in NS 101 SUCourse shows the magnitude and direction of the Coulomb force between two charges for different positions of the charge q.

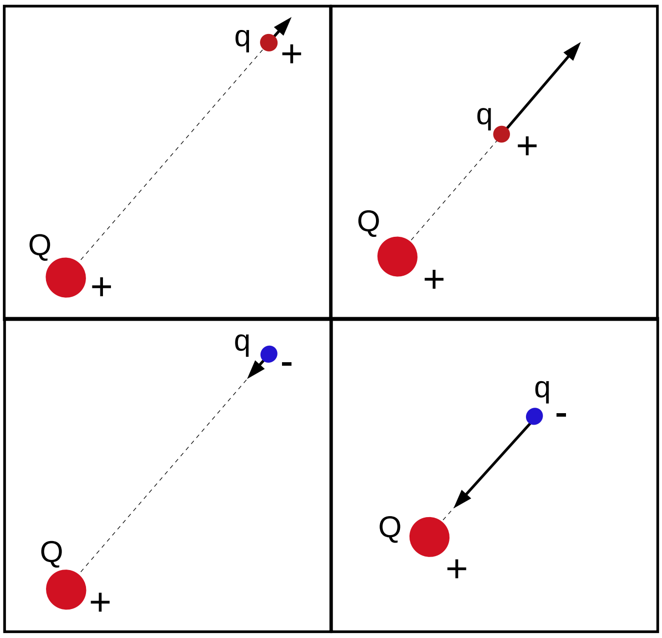


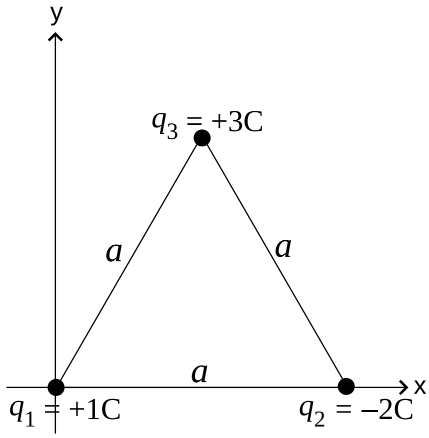
Figure 16.1:

The magnitude of the force (called electrostatic force or Coulomb force) is inversely proportional to the square of the distance between the two charges. By using different charges in the experiment one can show that the force is just proportional to either charge. Thus,

$$\mathbf{F}_E = k\frac{qQ}{r^2}\hat{\mathbf{e}}_\mathbf{r} \tag{16.1}$$

where r is the distance between the two charges, and $\hat{\mathbf{e}}_\mathbf{r}$ is the unit vector indicating the radial direction pointing from Q towards q. This is the direction of the force that Q applies on q. The constant in front is given in SI units as $k \equiv 1/(4\pi\epsilon_0) = 8.98 \times 10^9\ N\ m^2\ C^{-2}$ where $\epsilon_0 = 8.85 \times 10^{-12}\ N^{-1}\ m^{-2}\ C^2$ is just another constant. It is convenient to express the proportionality constant k in terms of 4π and another constant ϵ_0. The reason for this will become clear later.

Solved Problem: Coulomb Force

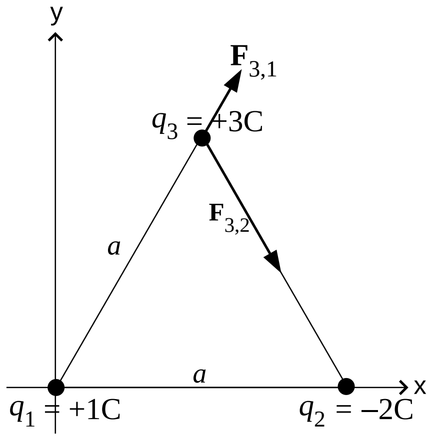


Three charges are placed at the corners of an equilateral triangle as shown in the figure. Find the force exerted on q_3 by the other two charges.

Coulomb force is a vector quantity, so we first need to determine the magnitude and direction of each force separately, then add all of the force vectors to find the total force.

Here, there are two forces exerted on q_3: $\mathbf{F}_{3,1}$ (due to q_1) and $\mathbf{F}_{3,2}$ (due to q_2).

To calculate the Coulomb force, we should first find the direction of the force by looking at the signs and the locations of the charges. Here, $\mathbf{F}_{3,1}$ pointing away from q_1 and $\mathbf{F}_{3,2}$ pointing towards q_2, as shown below.



Continues on next page...

...Continued from previous page

Once the directions are clear, we can find the magnitudes of these forces using Equation 16.1:

$$|\mathbf{F}_{3,1}| = k\frac{|q_3||q_1|}{a^2} = k\frac{(3C)(1C)}{a^2}\ N$$
$$|\mathbf{F}_{3,2}| = k\frac{|q_3||q_2|}{a^2} = k\frac{(3C)(2C)}{a^2}\ N$$

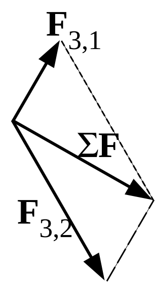
Notice that the magnitude is always positive.

We can now combine the directions and magnitudes and write $\mathbf{F}_{3,1}$ and $\mathbf{F}_{3,2}$ as vectors, using geometry:

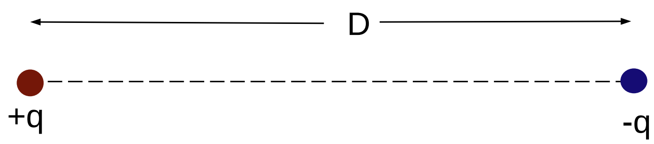
$$\mathbf{F}_{3,1} = k\frac{3}{a^2}[\cos 60°\mathbf{i}\ + \sin 60°\mathbf{j}\ ] = k\frac{3}{a^2}[\frac{1}{2}\mathbf{i}\ + \frac{\sqrt{3}}{2}\mathbf{j}\ ]\ N$$
$$\mathbf{F}_{3,2} = k\frac{6}{a^2}[\cos 60°\mathbf{i}\ - \sin 60°\mathbf{j}\ ] = k\frac{6}{a^2}[\frac{1}{2}\mathbf{i}\ - \frac{\sqrt{3}}{2}\mathbf{j}\ ]\ N$$

$$\text{So,}\quad \Sigma\mathbf{F} = \mathbf{F}_{3,1} + \mathbf{F}_{3,2} = k\frac{3}{a^2}[(\frac{1}{2}+1)\mathbf{i}\ + (\frac{\sqrt{3}}{2} - \sqrt{3})\mathbf{j}\ ]$$
$$= k\frac{3}{a^2}[\frac{3}{2}\mathbf{i}\ - \frac{\sqrt{3}}{2}\mathbf{j}\ ]\ N$$

We can also use geometry to show the direction of $\Sigma\mathbf{F}$:



Consider equal and opposite charges $+q$ and $-q$ separated by a distance D, as shown in Figure 16.2. Such an arrangement is called an ***electric dipole***. The electric dipole is the



simplest form of charge separation in neutral objects. It is very common in Nature. Many diatomic molecules are electric dipoles because the electrons in the molecule are concentrated towards one of the atoms, turning it into a negative ion, and the other atom into a positive ion. The two atoms turned into ions attract each other, but cannot accelerate all the way into each other because of repulsive forces from all the electrons. The forces come to balance at some equilibrium separation D of the atoms. The NaCl, table salt, molecule is an example of an electric dipole, forming an *ionic bond.* A macroscopic example of an electric dipole is a glass rod whose opposite ends have been charged by rubbing the rod against cloth.

16.3 The Electric Field and Voltage

A charge Q applies force on *any* other charge.

> Charge Q gives space a property such that when any charge q is placed at some distance from Q, there is a force exerted on it.

This property of space that acts as the intermediary of the force acting at a distance is called a *field*, in the present case an *electric field.*

The force on a charge q placed at the point $\mathbf{r}$ can be written as

$$\mathbf{F}_E = qk\frac{Q}{r^2}\hat{\mathbf{e}}_\mathbf{r} \equiv q\mathbf{E}(\mathbf{r}) \tag{16.2}$$

defining the electric field $\mathbf{E}(\mathbf{r})$ at the point at position $\mathbf{r}$ from the charge Q:

$$\mathbf{E}(\mathbf{r}) = k\frac{Q}{r^2}\hat{\mathbf{e}}_\mathbf{r}. \tag{16.3}$$

The electric field is a vector quantity. Its unit is Newton/Coulomb. [4]

For a single point charge Q at the origin, the electric field set up by Q is always along the radial direction $\hat{\mathbf{e}}_\mathbf{r}$, and its magnitude decreases as r becomes larger. If Q is positive, the field is radially outward, and if negative, the field points radially inward.

By joining the electric field vector arrows at different points, one obtains electric field *lines* and an electric field *pattern*, which is very useful for understanding the overall properties of an electrical system. The electric field lines are closer together where the electric field is stronger, and apart from each other where the electric field is weaker, as you can observe in the electric field pattern of a single point charge in the below figure. Electric field lines converge onto places containing negative charges, and diverge from places containing positive charges.

The force $\mathbf{F}_E$ that Q applies on a particular q placed at $\mathbf{r}$ is the Coulomb force:

$$\mathbf{F}_E = q\mathbf{E} = k\frac{qQ}{r^2}\hat{\mathbf{e}}_\mathbf{r} \tag{16.4}$$

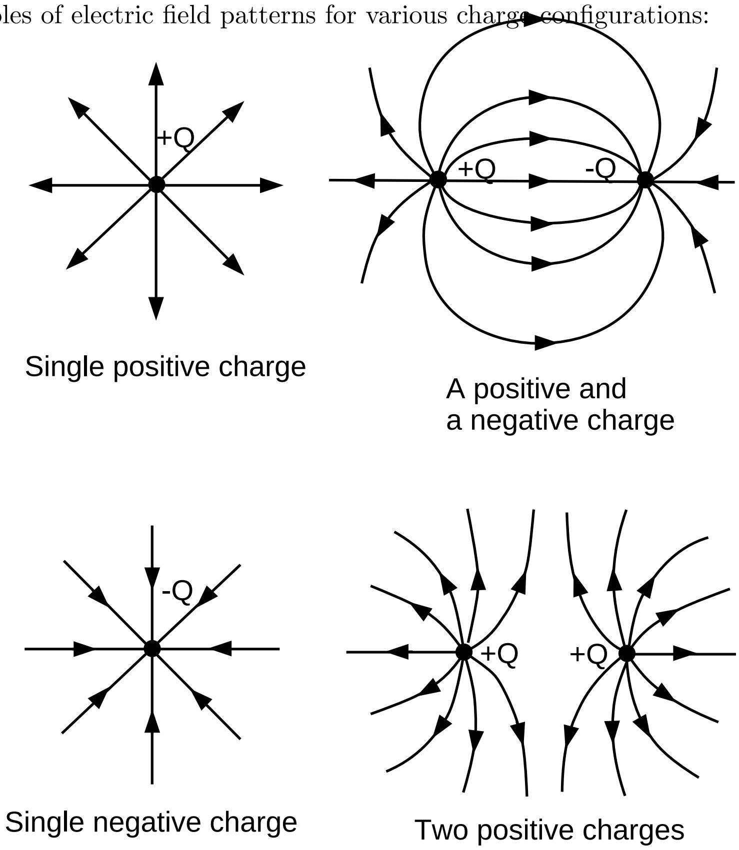
Having defined the the force on a charge q as the electric field times q, $\mathbf{F}_E = q\mathbf{E}$ ($\mathbf{E} = \mathbf{F}_E/q$), it is natural to extend the concept to the work done by the electric force on q. As it turns out, the electrostatic force is a conservative force, so there is a potential energy is associated with it. Thus *potential energy of a charge q is proportional to the charge q*:

$$W = \int \mathbf{F}_E \cdot \mathbf{dr} = q\int \mathbf{E} \cdot \mathbf{dr} = -\Delta U_E. \tag{16.5}$$

[4] If $q = 1Coulomb$, a unit test charge, you can look at the simulation in NS 101 SUCourse again as depicting the electric field.

Electric Field Patterns

Examples of electric field patterns for various charge configurations:



Now define

$$\int \mathbf{E} \cdot \mathbf{dr} = -\Delta V \quad \text{"}\textbf{Voltage Difference}\text{"} \tag{16.6}$$

Thus $W = -q\Delta V$, and $\Delta U = q\Delta V$, leading to $U(\mathbf{r}) = qV(\mathbf{r})$.

The voltage (or, as it is sometimes called, "the electric potential") at each point $\mathbf{r}$ is defined such that the electrostatic potential energy is just the charge q times the voltage at that point. But the voltage is there, due to all other "source" charges, even when there is no charge at point $\mathbf{r}$. The voltage at $\mathbf{r}$ is set up by charges at other places – *just like the electric field* $\mathbf{E}$*, voltage is a property of space!* The relation between electric field and voltage is:

$$E_x = -\frac{dV}{dx} \quad \text{in one dimension, and} \tag{16.7}$$

$$\mathbf{E} = -\left(\frac{\partial V}{\partial x}\,\mathbf{i} + \frac{\partial V}{\partial y}\,\mathbf{j} + \frac{\partial V}{\partial z}\,\mathbf{k} \right) = -\nabla V \quad \text{in general (3 dimensions)} \tag{16.8}$$

The dimension of voltage is $[V] = [U/q]$ so its unit is Joule/Coulomb, which is given the name "Volt" in honour of Alessandro Volta.

$$1 \text{ Volt} = 1 \text{ Joule/Coulomb} = 1 \text{ N m/Coulomb}$$

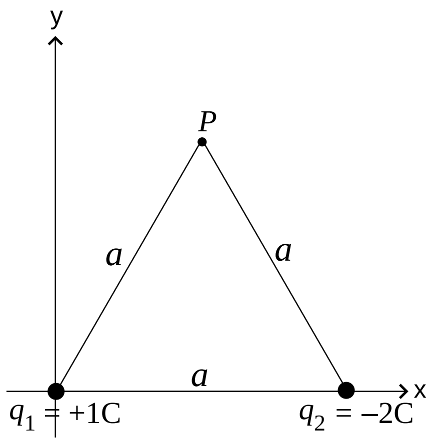
The unit of the electric field is

$$1 \text{ N/Coulomb} = 1 \text{ Volt/m}$$

Since the electric field due to Q is just proportional to Q ("the electric field is linear in Q"), the total electric field due to two separate charges Q_1 and Q_2 is simply the sum of the electric field due to Q_1 and the electric field due to Q_2. This simple but important observation, true for all *linear* situations, is called "*The Superposition Principle*".

Coulomb's Law leads to *Gauss' Law*, the first one of the *Maxwell Equations*. Gauss' Law formulates a powerful connection between the global electric field distribution and the charges that are the source of that electric field.

Solved Problem: Electric Field and Electric Potential



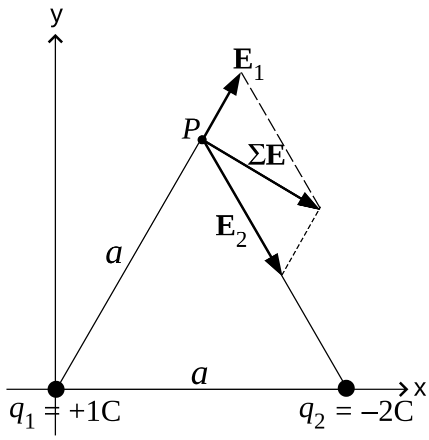
Consider the three charges on the equilateral triangle in the previous example, without q_3 at a point P (see the figure).

(a) Find the electric field at P.

An electric field is also a vector quantity, defined as "the force on a unit positive charge", so we need to place a positive test charge $+q_t$ at P to find the direction of the electric field at P. Since $\mathbf{E} = \mathbf{F}_E/q$, the directions of the electric fields are the same as the directions of the Coulomb forces on $+q_t$ due to q_1 and q_2 (see the figure below).

Continues on next page...

...Continued from previous page



Using Equation 16.3, the magnitudes of $\mathbf{E}_1$ and $\mathbf{E}_1$ are:

$$|\mathbf{E}_1| = k\frac{|q_1|}{a^2} = k\frac{1}{a^2}\ N/C$$
$$|\mathbf{E}_2| = k\frac{|q_2|}{a^2} = k\frac{2}{a^2}\ N/C$$

We can also find these from the fact that $\mathbf{E}_1 = \mathbf{F}_{q_t,1}/q_t$ and $\mathbf{E}_1 = \mathbf{F}_{q_t,1}/q_t$. Similar to the previous Coulomb Force example, we find the total electric field at P with a vector addition, $\Sigma\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$:

$$\mathbf{E}_1 = k\frac{1}{a^2}[\cos 60°\mathbf{i}\ + \sin 60°\mathbf{j}\ ] = k\frac{1}{a^2}[\frac{1}{2}\mathbf{i}\ + \frac{\sqrt{3}}{2}\mathbf{j}\ ]\ N/C$$
$$\mathbf{E}_2 = k\frac{2}{a^2}[\cos 60°\mathbf{i}\ - \sin 60°\mathbf{j}\ ] = k\frac{2}{a^2}[\frac{1}{2}\mathbf{i}\ - \frac{\sqrt{3}}{2}\mathbf{j}\ ]\ N/C$$

$$\text{So,}\quad \Sigma\mathbf{E} = \frac{k}{2a^2}[3\mathbf{i}\ - \sqrt{3}\mathbf{j}\ ]\ N/C \quad \text{(in the direction shown above)}$$

(b) Find the electric potential (or voltage) at P.

The electric potential, V, is a scalar quantity, and is found by

$$V = -\int \mathbf{E}\cdot\mathbf{dr} = \frac{kq}{r}$$

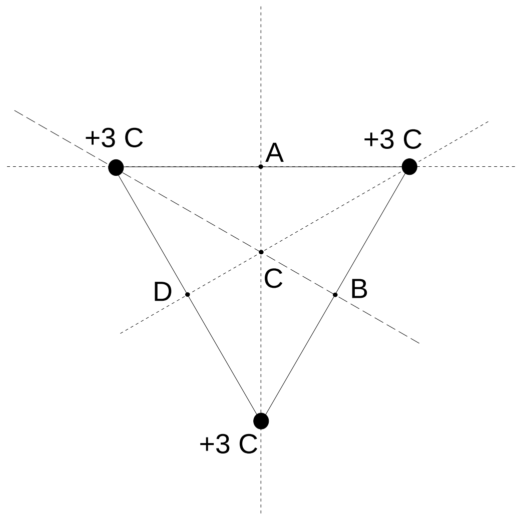
In this case, the total electric potential is just a scalar sum of the potential due to q_1 and q_2, so:

$$V = \frac{kq_1}{r_1} + \frac{kq_2}{r_2} = \frac{1k}{a} + \frac{-2k}{a} = -\frac{k}{a}\quad Volt$$

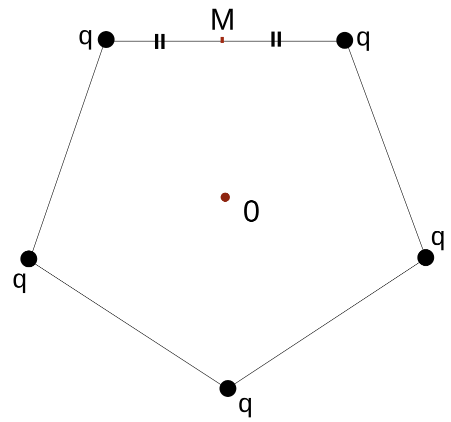
Notice that we kept the signs of the charges.

CHAPTER 16 - PROBLEMS:

1. (a) How many electrons are needed to make up a total charge of $-1\,\mathrm{C}$?
   (b) How many atoms are there in a typical macroscopic object of centimeter dimensions, as an order of magnitude?
   (c) If you rub your plastic comb until one side of it has a negative charge of $1\,\mathrm{C}$, what fraction of the atoms in the comb have contributed an electron to this polarization?
2. What are the differences and resemblances between the gravitational force and the Coulomb's force?
3. Calculate the force between an electron and a proton in a Hydrogen atom. Compare it to the gravitational force between these two particles. The orbital radius of the H atom is 0.5 Angstrom. Mass of an electron : 9.1×10^{-31} kg, mass of a proton: 1.7×10^{-27} kg, 1 Angstrom $= 10^{-10}$ m.
4. Charges of + 2 C are placed at points with (x, y) coordinates (0, 1 m) and (0, -1 m), and charges of - 2 C are placed at points (1 m, 0) and (-1 m, 0). Such a charge distribution is called a "*quadrupole*".
   (a) Make a figure showing the quadrupole, and sketch the electric field pattern.
   (b) What is the magnitude and the direction of the electric field at the origin (0, 0).
   (c) Find the magnitude and direction of the electric field at a point of your choice, other than the origin.



5. Charges of $+3\ C$ are at the corners of an equilateral triangle of side length 1 m as shown in Figure 4. What is the magnitude and direction of the electric field at points A, B, C and D ?
6. Sketch the electric field pattern of the three-charge configuration in Problem 5, shown in Figure 4.



7. Equal charges $q > 0$ are placed at each corner of a pentagon as shown in Figure 6.

   (a) What is the direction of the electric field at point M?

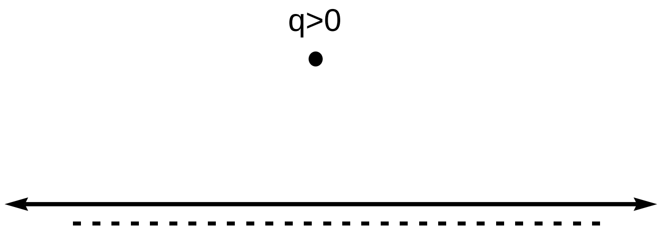
   (b) What is the magnitude and direction of the electric field at point 0?

8. Consider a ring of a large number N of equal positive charges, equally spaced on the ring.

   (a) What is the magnitude of the electric field at the center of the ring?

   (b) What is the direction of the electric field at a point on the ring?

   (c) What is the direction of the electric field at a point on the ring if the charges are all equal and negative?



9. Sketch the "*lines of force*", the electric field configuration, for a positive charge next to a uniformly negatively charged infinite plane as seen in Figure 8.

10. A point particle of positive charge q and mass m is thrown vertically between two oppositely charged horizontal infinite plates separated by a distance d as shown in Figure 16.2. The electric field $\mathbf{E}$ is uniform, and directed perpendicularly to the plates, pointing from the positively charged plate towards the negatively charged plate, as we shall discuss in "Chapter 18". If the strength of the uniform electric field is E_0, what is the minimum initial speed v_i for the particle to reach the upper plate? How long does it take to travel the distance d? Ignore gravity.

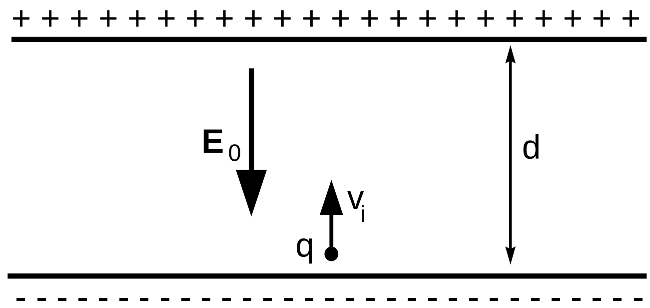


Figure 16.2:

11. If the particle in the previous problem has an initial velocity $\mathbf{v}_i$ which is not vertical but makes some angle θ to the plates, what kind of trajectory does it follow? Ignore gravity.

12. An electron is placed in a vertical *cathode ray tube*, like the tube behind your TV screen, such that the electric field, of strength E, is in the vertically downward direction.

    (a) What is the direction of the electric force on the electron?

    (b) What is the direction and the magnitude of the Earth's gravitational force on the electron?

    (c) What must be the strength E of the electric field, in vacuum, that can balance the weight of the electron?

    (d) With this electric field, what is the voltage difference between the two ends of the cathode ray tube? The length of the tube is 50 cm. Which end is at the higher voltage, the (+) charged end or the (−) charged end?

Chapter 17

Gauss' Law

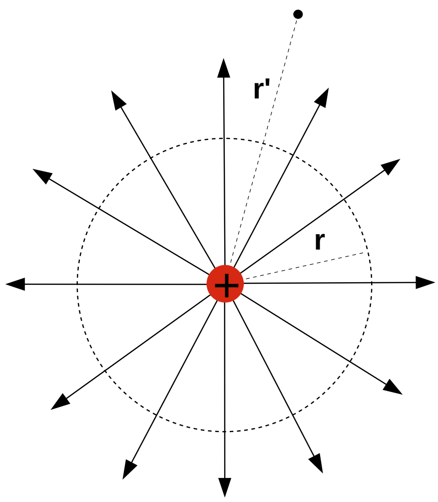


Figure 17.1:

In Figure 17.1 you see the electric field lines set up by a positive point charge Q. The electric field is in the radially outward direction, and has the form

$$\mathbf{E} = \frac{kQ}{r^2}\hat{\mathbf{r}} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}\ N\ C^{-1}. \tag{17.1}$$

In the SI system the unit of electric field is $N\ C^{-1}$, equivalently defined as a Volt / meter, V/m.

At all points on a sphere of radius r centered on the charge the electric field has the same value $E = Q/4\pi\epsilon_0 r^2$, and is perpendicular to the spherical surface whose area is $A = 4\pi r^2$. If we calculate the product EA, the factor $4\pi r^2$ cancels out. Thus

$$EA = \frac{Q}{\epsilon_0} \tag{17.2}$$

Independently of its radius, the product of the area of any sphere centered on the charge and the value of the electric field on that sphere gives the charge, up to a constant depending on the system.

This statement can be generalized:

The sum of : (pieces of area of a closed surface) $\times$ (the component of electric field perpendicular to each small area element in the outward perpendicular direction) equals a constant times the total charge inside the surface, no matter what the shape of the surface is:

$$\oint \mathbf{E} \cdot \mathbf{dS} = \frac{Q_{\text{in}}}{\epsilon_0} \qquad [\textbf{Gauss}' \textbf{ Law}] \tag{17.3}$$

The little circle around the integral sign denotes integration over *a closed surface*. Here $\mathbf{dS}$ is a vector whose magnitude is the area element dS and whose direction is perpendicular to that area element in the outward direction with respect to the closed surface. Thus the scalar product $\mathbf{E} \cdot \mathbf{dS}$ is the product of the area of that element and the component of the electric field vector perpendicular to that area in the outward direction.

The scalar product $\mathbf{E} \cdot \mathbf{dS}$ is the "*flux* of the electric field outward through the area element dS".

A *positive* charge is the *source* of outward net flux of electric field in the space around the charge. A *negative* charge sets up an electric field with a net inward flux through any closed surface enclosing the negative charge. A *negative* charge is a *sink* for the electric field.

Flux: How to calculate flux?

If the area is oriented perpendicular to the direction of the electric field, the *flux*, $\mathbf{E}{\cdot}\mathbf{dS}$ is maximum, and equals EdS, the product of the magnitude of the electric field and the area. If the surface is oriented parallel to the electric field, the electric field is not through the surface, so the flux of the water through the area is zero. This is seen in the expression $\mathbf{E} \cdot \mathbf{dS} = 0$: Note that when the area is parallel to $\mathbf{E}$, the vector $\mathbf{dS}$, which is perpendicular to the surface, is perpendicular to $\mathbf{E}$ also, and $\mathbf{E}{\cdot}\mathbf{dS} = 0$ indeed.

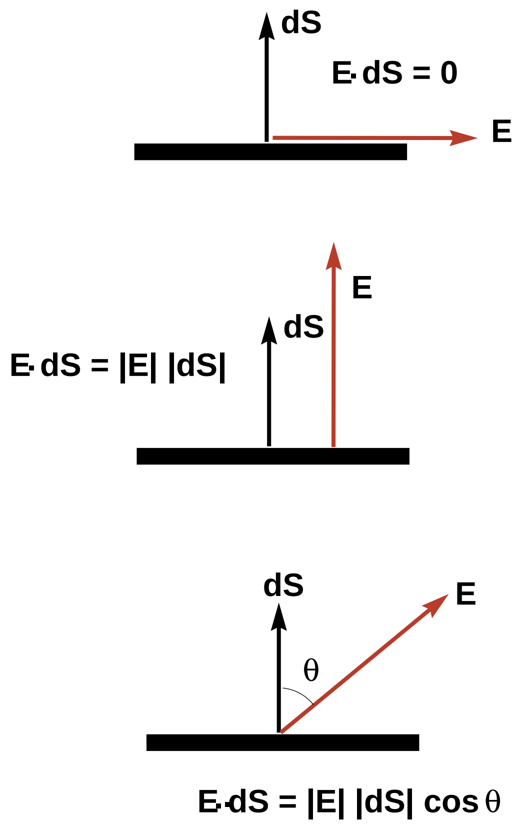
If the surface is neither parallel nor perpendicular to $\mathbf{E}$ but is at some angle θ between the field $\mathbf{E}$ and the surface normal (perpendicular) direction $\mathbf{dS}$, then the flux $\mathbf{E} \cdot \mathbf{dS} = E\, dS \cos\theta$, a value intermediate between 0 and $E\, dS$ as seen in the figure below.

Now imagine a porous cage, a closed surface made up from many small surface elements $\mathbf{dS}$.

- If $\oint \mathbf{E} \cdot \mathbf{dS} > 0$, the flux summed over the closed surface, is positive, then there is a net flux of electric field *out of* the cage: there must be a *source* of electric field enclosed within the cage.
- If $\oint \mathbf{E} \cdot \mathbf{dS} < 0$, then there is net electric field flux *into* the cage: electric field is pointing into the cage from all around – it is disappearing inside the cage, so there is a *sink* inside the cage.

Continues on next page...

...Continued from previous page



Solved Problem: Calculating Flux

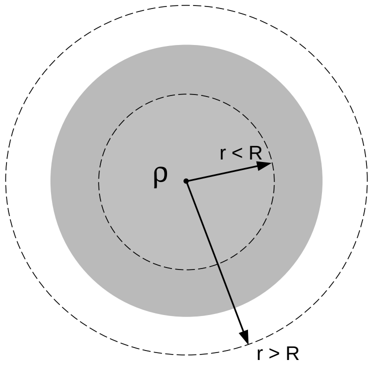
If the electric field is represented with $\mathbf{E} = -3\hat{\mathbf{i}} + 4\hat{\mathbf{j}}$ N/C, how much flux goes through an area of 2 m^2 on the xy plane? How about through the same area in the xz plane?

The electric field is on the xy plane. When the area is on the xy plane the electric field $\mathbf{E}$ is perpendicular to the area vector dS then the flux through this area is zero.

However, when the area is on the xz plane, the area vector is in y direction. In that case, the flux is

$$\oint \mathbf{E} \cdot \mathbf{dS} = \oint [-3\hat{\mathbf{i}} + 4\hat{\mathbf{j}} N/C] \cdot 2\hat{\mathbf{j}} m^2 = 8Nm^2/C$$

Solved Problem:Electric field of a charged insulating sphere



A sphere of radius R made of an *insulating* material has a uniform charge density ρ.

(a) Find the magnitude of the electric field at a distance $r < R$, using Gauss' Law.

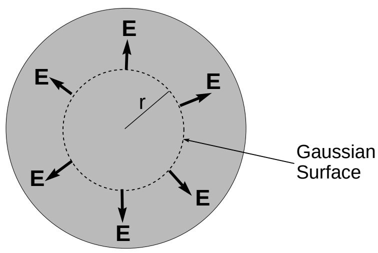
The law states that:

$$\oint \mathbf{E} \cdot \mathbf{dS} = \frac{Q}{\epsilon_0} \tag{17.4}$$

This means we can choose any closed surface in the charged sphere and the electric flux through this sphere is proportional to the total charge enclosed in the sphere.

This statement is quite strong as it is, and it is true for any closed surface. However if we are clever enough to use symmetries in the problem and choose a surface such that the electric field on the surface is perpendicular to the surface at every point and its magnitude is the same at every point on the surface, then we can calculate the electric field by using Gauss' law very easily.

So, let us choose a Gaussian surface that is spherical and concentric with the insulating sphere in the problem, as shown in figure below.



Since the charge distribution is uniform, on this Gaussian sphere at every point the electric field is constant due to symmetry. It is also pointing radially outward, that is, it is perpendicular to the Gaussian surface at every point on the surface.

Continues on next page...

...Continued from previous page

Gauss' Law then gives:

$$E(r) \oint dS = \frac{Q_{in}}{\epsilon_0} \tag{17.5}$$

The integral is $\oint dS = 4\pi r^2$, and the charge in the Gaussian surface can be obtained by multiplying the charge density by the volume enclosed by the Gaussian surface

$$Q_{in} = \rho \frac{4}{3} \pi r^3$$

.

Substituting Q_{in} and $\oint dS$ in Equation 17.5, we find the electric field to be:

$$E(r) = \frac{\rho r}{3\epsilon_0} \tag{17.6}$$

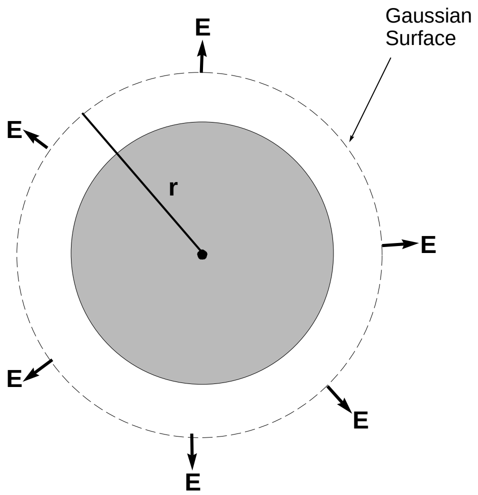
What is the total charge on this sphere?

(b) Since charge is uniformly distributed, the total charge on the sphere is obtained by multiplying the charge density by the total volume of the sphere:

$$Q = \rho \frac{4}{3} \pi R^3$$

(c) Find the magnitude of the electric field at a distance $r > R$, using Gauss' Law.

In order to find the electric field outside of the sphere, we follow the same footsteps as (a) with a Gaussian surface shown in the below Figure.



The Gauss' law can be written for this surface as:

Continues on next page...

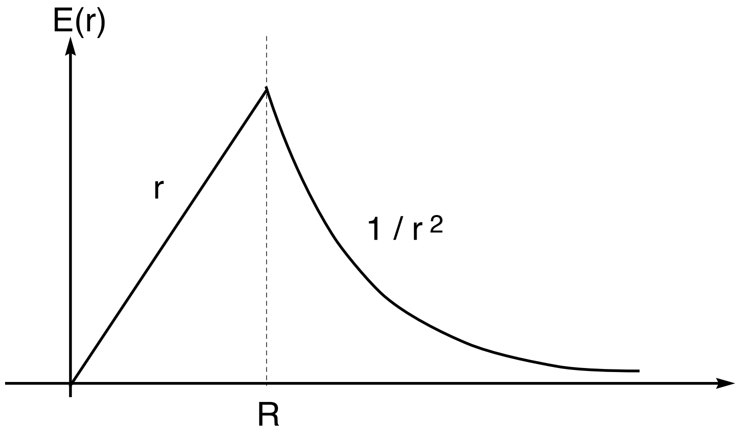
...Continued from previous page

$$E(r)4\pi r^2 = \frac{4\pi\rho R^3}{3\epsilon_0}$$

So the electric field outside of the insulating sphere is:

$$E(r) = \frac{\rho R^3}{3\epsilon_0 r^2}$$

Sketch the magnitude of the electric field $E(r)$ from $r = 0$ to $r > R$.



(d)

CHAPTER 17 - PROBLEMS:

1. The uniform electric field shown in Figure 17.2 is $\mathbf{E} = 5\hat{\mathbf{j}}$ N/C everywhere. The cube has 5 cm edges.
   (a) What is the total electric flux outward through the closed surface of the cube? The flux integral
   $$\Phi = \oint \mathbf{E} \cdot \mathbf{dS}$$
   is taken over all six faces?
   (b) Can you think of any closed surface through which the flux is not zero?

2. A sphere of radius $R = 10\ cm$ made of insulating material has a uniform charge density $\rho = 100\ C\ m^{-3}$ as seen in Figure 17.3.
   (a) What is the value of the electric field at $r = 5\ cm$?
   (b) What is the value of the electric field at $r = 10\ cm$? At $r = 20\ cm$?
   (c) Graph the magnitude of the electric field $E(r)$ from $r = 0$ to $r = 20$ cm.

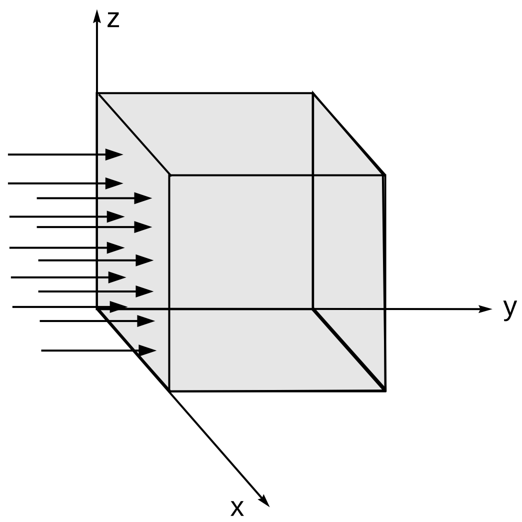


Figure 17.2:

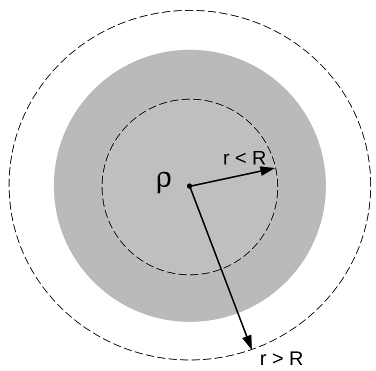


Figure 17.3:

(d) Take the voltage at $r = 0$ to be 0 volts (this point is called the "ground"). Find the voltage $V(r)$ at any r by integrating the electric field $E(r)$ along the radial direction.

(e) Graph the voltage $V(r)$ from $r = 0$ to $r = 20$ cm.

3. A point charge q = $+5$ C is at the center of a thin *conducting* spherical shell of radius 1 m, carrying a total charge -5 C as seen in Figure 17.4.

   (a) The charge on a conductor is spread out at uniform charge density per unit area on the outer surface of the conductor (why?). What is the charge density on the spherical shell σ in C/m^2 ?

   (b) What is the magnitude and direction of the electric field $\mathbf{E}$(r), at all distances r from the origin where the positive charge is.

4. An infinite straight wire has a charge density $\lambda = -2C/m$ all along its length as seen in Figure 17.5. Use Gauss' Law on the closed cylinder shown in the figure to find the

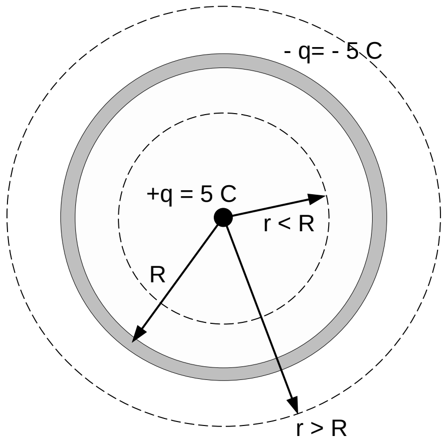


Figure 17.4:

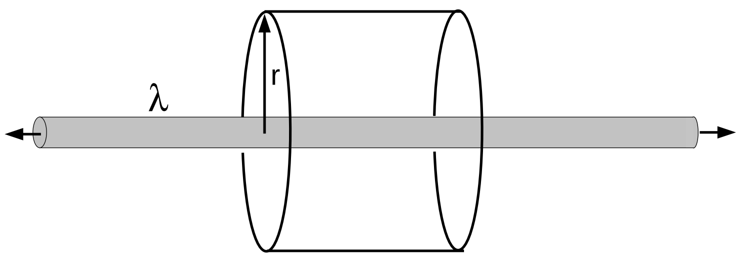


Figure 17.5:

magnitude and direction of the electric field at distance r from the wire. Evaluate for $r = 1\ m$.

5. You bring a negative point charge $-q$ to a distance r away from a positive charge $+Q$, with a constant speed from a very far away distance (= infinity, where $V(\infty) = 0$), as shown in Figure 24.4.

   (a) What is the work done by you?

   (b) What is the work done by the charge $-q$? How much energy does it gain or lose?

   (c) Find the voltage at r due to $+Q$, using the work found above.

   (d) Find the electric field at r due to $+Q$, using the voltage found above. Does this agree with the electric field found using the Coulomb force?

Chapter 18

Parallel Plate Capacitors

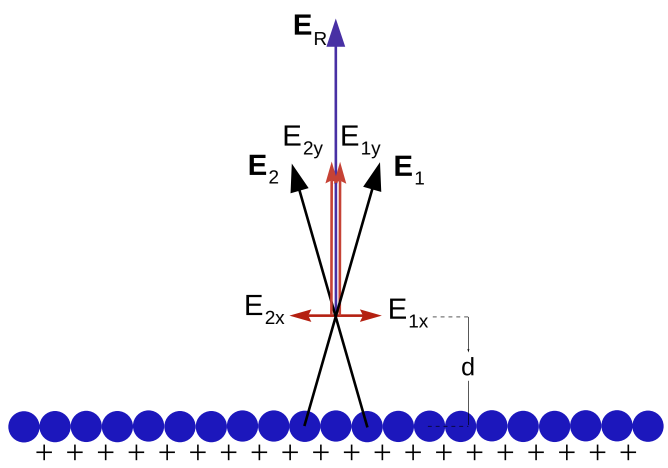


Figure 18.1:

In Figure 18.1 there is a row of charges on a straight line. This represents a plane of charges in cross section. The question is:

What kind of electric field do these charges set up?

Let us approximate the line and plane as if they extended to infinity. This approximation simply means that the point where the electric field is measured is far from the edges compared to its distance from the line or plane. Such a point can be taken to be above the *middle* of the infinite line or plane: very far parts of the line do not contribute significantly, and for nearer parts there are as many charges to the right as there are to the left, and as many to the front as there are to the back of the point[1]. The electric field contributions of the symmetric pair of charges are shown as $\mathbf{E}_1$ and $\mathbf{E}_2$ in Figure 18.1. The components of the electric fields parallel to the line or plane (E_{1x} and E_{2x}) cancel, while the perpendicular components (E_{1y} and E_{2y}) add up.

Doing this for all symmetric pairs of charge elements we find that the net electric field should be perpendicular to the plane. The point where we are calculating the field is an arbitrary point; there is nothing special about it. By symmetry, at all points above and below all parts of the "*infinite*" plane, that is, at points for which the distance from the plane

[1]The parallel plate capacitor simulation on the NS 101 SUCourse site shows the electric field of a row of charges. If you click on a charge element in the line, the simulation shows the electric field due to that charge and its partner at the symmetric location on the other side of the "midpoint" where the field is measured.

is much less than the distance from the edges, the electric field is directed perpendicular to the plane.

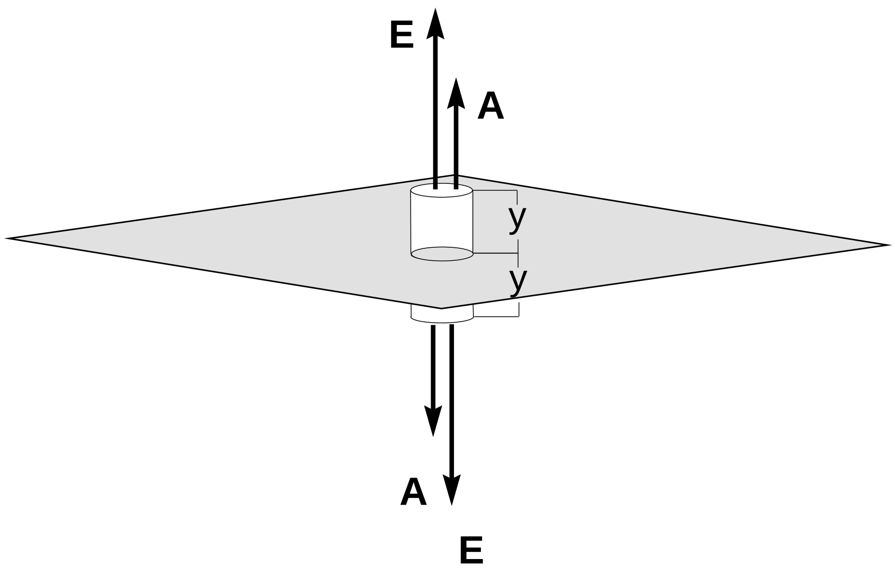


Figure 18.2:

Now let us calculate the magnitude of this electric field using Gauss' Law (Equation 17.3). To find the electric field at a distance y from the plane using Gauss' law, *any* closed surface can be used. It is practical to choose surfaces for which the flux calculation is easy.

We make use of symmetry to choose the pieces of the closed surface such that the electric field is either perpendicular or parallel to the surface. In the present example the easy choice is a cylinder (or prism) with top and bottom surfaces parallel to the infinite charged plane, and side surfaces perpendicular to the plane.

Form a closed surface with an area A parallel to the plane of charges, at distance y above the plane, and another parallel plane of area A at a distance y below the plane of charges as seen in Figure 18.2. These two planes form the top and bottom of our closed surface, closed up with side surfaces perpendicular to the plane of charges. On the top and bottom, the electric field flux is positive and equals EA, where E is the magnitude of the electric field, since the field is perpendicular to these surfaces and therefore parallel to dS. On the side surfaces the electric flux is zero since the electric field, being perpendicular to the plane of charges, is parallel to the side surfaces. Thus the total flux through the closed surface is:

$$\oint \mathbf{E} \cdot \mathbf{dS} = 2EA \tag{18.1}$$

Assume that the plane has a surface charge density (charge per area) of σ. By Gauss' Law, the total flux found above must equal a constant $1/\epsilon_0$ times the charge inside the enclosed piece of the plane, so:

$$\oint \mathbf{E} \cdot \mathbf{dS} = 2EA = \frac{Q_{\text{enc}}}{\epsilon_0} = \frac{\sigma A}{\epsilon_0}, \tag{18.2}$$

and therefore,

$$E = \frac{\sigma}{2\epsilon_0} \tag{18.3}$$

This is the magnitude of the uniform electric field due to an "infinite" plane of positive charge. The field is perpendicular to the charged plane, and pointing away from the plane. Since the field does not depend on y, it has the same value at all points on either side of the plane of charge.

A *capacitor* is any system that can hold positive or negative charges or both kinds of charge separated from each other.

Now consider a capacitor, made of a pair of parallel conducting plates planes with positive charge on one and negative charge on the other (see Figure 18.3).

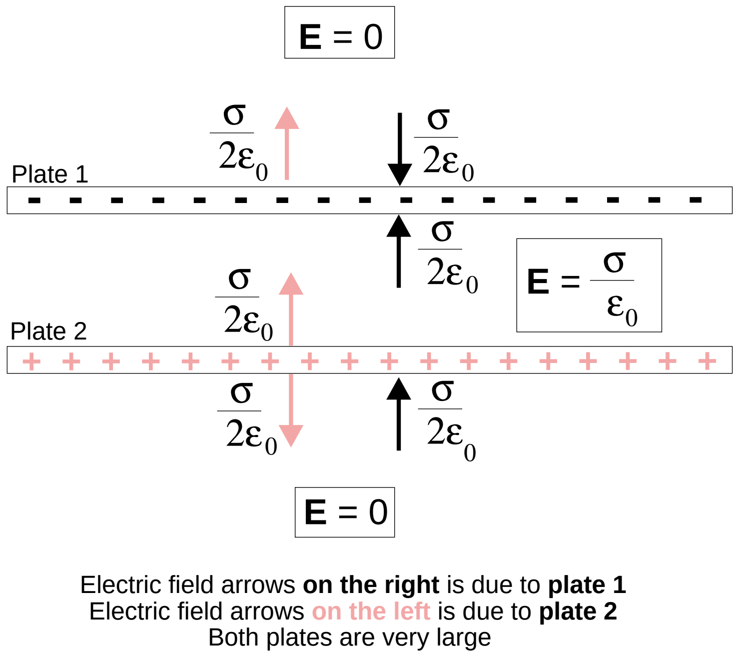


Figure 18.3:

The field due to the negatively charged plate is uniform and perpendicular to the plate pointing *towards* the negatively charged plate from either side. The field due to the positively charged plate is also uniform and perpendicular to the plate, pointing *away* from the positively charged plate on either side.

As we saw in our discussion of Coulomb's Law and Gauss' Law, the total electric field due to different charges is the sum of the electric fields contributed by the individual charges. This "*superposition principle*" holds because the electric field of any charge is simply proportional to the charge ("*linear* in the charge") and NOT to some nonlinear function of charge like q^3 or $1/q^5$.

For the capacitor, the fields of its positively charged and negatively charged planes are *superposed*. The total field is zero outside the capacitor. In between the parallel plates, the field points perpendicularly to the capacitor plates, in the direction away from the positively charged plane and towards the negatively charged plane. Since each plate provides $E = \sigma/2\epsilon_0$, the magnitude of the total electric field is

$$E = \frac{\sigma}{\epsilon_0}. \tag{18.4}$$

The voltage difference between the plates is the integral of the electric field from the negative to the positive plate. In this case, the electric field is constant, so if the plate separation is D, the voltage difference is:

$$V = ED = \frac{\sigma}{\epsilon_0} D = \frac{QD}{\epsilon_0 A} \equiv \frac{Q}{C} \tag{18.5}$$

Because the electric fields are linear in the source charge, Q, voltages are also proportional to the source charges. The proportionality constant is defined as $1/C$, where C, called the "*capacitance*, depends on the geometry of the system. In the present case, the parallel plate capacitor,

$$C = \frac{\epsilon_0 A}{D} \tag{18.6}$$

The SI unit of capacitance in is the *Farad*, named after Fraday.

$$1 \text{ Farad} \equiv 1 \text{ Coulomb/Volt} = 1 \text{ (Coulomb)}^2\text{/Joule}$$

.

CHAPTER 18 - PROBLEMS:

1. A parallel plate capacitor has circular plates of radius R separated by a distance $D \ll R$. Make a picture of the capacitor in cross section, on a plane through the central axis of the capacitor. This will represent the two plates as two parallel lines of length $2R$. On this figure show the electric field lines deep inside the capacitor far from the edges, and also near the edges.

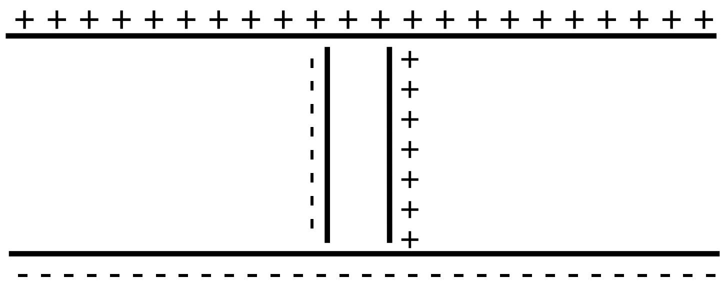


Figure 18.4:

2. Two parallel plate capacitors are placed one inside the other as shown in Figure 18.4. The plates do not touch each other. Draw the electric field lines.

3. A parallel plate capacitor has area $A = 100\ mm^2$ and separation $D = 3\ mm$. The plates carry charges of $+0.01\ C$ and $-0.01\ C$. Take $\epsilon_0 \cong 9 \times 10^{-12}\ C^2\ m^{-2}\ N^{-1}$.
   (a) What is the value of the capacitance in Farads?
   (b) What is the charge density σ?
   (c) What is the magnitude of the electric field between the capacitor plates, far from the edges?
   (d) What is the voltage difference between the plates?

4. A parallel plate capacitor with area A has surface charge density of $+\sigma$ and $-\sigma$, separated by distance d. If the distance d is increased, what will happen to:
   (a) the capacitance C?
   (b) the voltage difference V?
   (c) the charge Q on the plates?
   (d) the electric field E between the plates?

5. Consider a conducting sphere of radius R carrying total charge Q.
   (a) What is the magnitude and direction of the electric field at a point outside the sphere, at distance $r > R$ from the center of the sphere?
   (b) What is the voltage difference between the surface of the sphere and a point very far away, at "infinity"? Take $V(\infty) = 0$.
   (c) What is the capacitance of this charged sphere?

Exam Problems - Gauss' Law

1. **[Fall 2009, Final]** A point particle with charge $+Q$ and mass m is suspended with a string near an infinite sheet of charge with charge density $+\sigma$ as shown in Figure 18.5. The mass is at rest (in equilibrium) at a constant angle θ.

   (a) Find the Electric Field, $\mathbf{E}$, due to the infinite sheet of charge using Gauss' law.

   (b) Draw the free body diagram for the point particle and identify all the forces.

   (c) Find the tension on the string T (you can give your answers in terms of $\sin\theta$, $\cos\theta$ or $\tan\theta$).

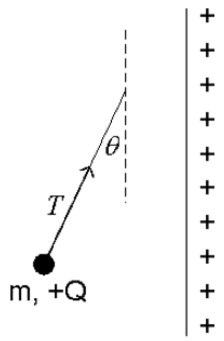


Figure 18.5:

2. **[Spring 2007, Final]** Figure 18.6 shows a charge of $+5\,C$ at the centre and two concentric, spherical conducting shells, of radii $2\,m$ and $4\,m$, carrying charges $+10\,Coulombs$ and $-20\,Coulombs$ respectively.
   Use Gauss' Law to find the magnitude and direction of the electric field at points P_1, P_2, P_3 and P_4 shown in the figure. The distances from the centre are: P_1: $1\,m$, P_2 : $3\,m$, P_3 and P_4 : $5\,m$. Express your answers in terms of $K = 1/(4\pi\epsilon_0)$ $-$ do not use the numerical value of K.

3. **[Fall 2006, Final]** An amount of positive charge Q is distributed uniformly inside a sphere of radius R (see Figure 18.7).

   (a) What is the charge density ρ inside this sphere?

   (b) Use Gauss' Law to find the magnitude of the electric field at a point P inside the sphere, $r < R$ (see the figure).

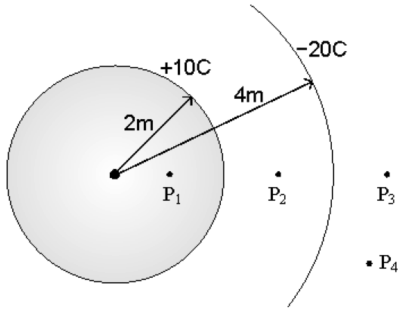


Figure 18.6:

(c) Use Gauss' Law to find the magnitude of the electric field at a point P'' outside the sphere, $r > R$ (see the figure).

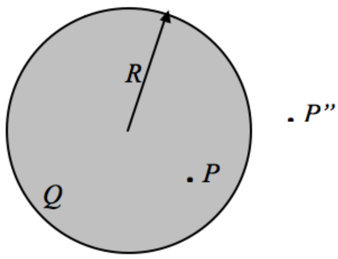


Figure 18.7:

4. **[Fall 2007, Final]** A solid insulating sphere with radius a and total charge Q uniformly distributed over its volume is surrounded by a concentric thin spherical shell with radius b and total charge $-Q$ uniformly distributed over its surface as shown in Figure 18.8. Using Gauss' Law, find the electric field $\mathbf{E}(r)$ (magnitude and direction)

   (a) for $r > b$

   (b) for $a < r < b$

   (c) for $r < a$.

   (d) What is the voltage difference between the two shells (between $r = a$ and $r = b$)?

5. **[Spring 2010, Final]** A solid sphere with uniform charge distribution has a radius $\mathrm{r_a}$ and volume charge density of ρ. There is a concentric spherical shell with uniform surface charge density of σ and radius $\mathrm{r_b}$ (see Figure 18.9).

   (a) Find the electric field $\mathbf{E}_1$ at P_1.

   (b) Find the electric field $\mathbf{E}_2$ at P_2.

   (c) Consider a point charge +q with mass m at P_2. What are the force $\mathbf{F}$ and acceleration $\mathbf{a}$ on the charge?

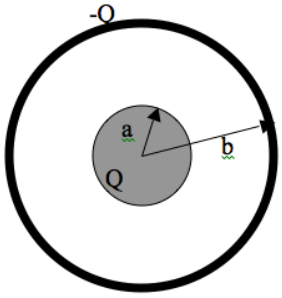


Figure 18.8:

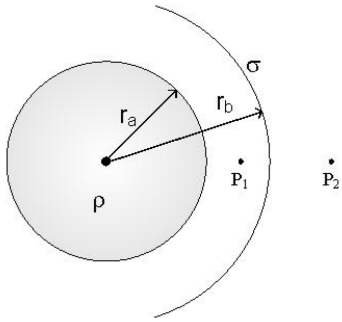


Figure 18.9:

[Remember: A vector has magnitude and direction]

6. **[Fall 2010, Final]** A charge $+2Q$ is spread uniformly across a solid sphere of radius a as shown in Figure 18.10 A charge $-Q$ is uniformly spread across a very thin concentric spherical shell of radius b. Use the Gauss' law to find the electric field vector $\mathbf{E}$ for

   (a) $r < a$

   (b) $a < r < b$

   (c) $r > b$

   (d) Sketch the Electric field as a function of r.

7. **[Spring 2011, Midterm 2]** An infinite straight wire has a charge density $\lambda = -2000\pi\epsilon_0$ Coulomb / m all along its length as seen in Figure 18.11.

   (a) What is the direction of the electric field on the cylinder − show the electric field vectors on the figure, at the points P1, P2 and P3 on the cylinder.

   (b) What is the flux of the electric field $\mathbf{E}$ through the cylindrical surface?

   (c) What is the flux of the electric field through the top and bottom surfaces?

   (d) Use Gauss' Law to find the electric field strength E at distance $r = 1\,\text{m}$ from the wire.

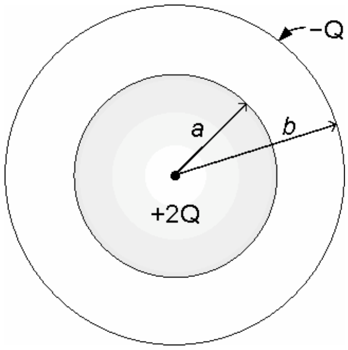


Figure 18.10:

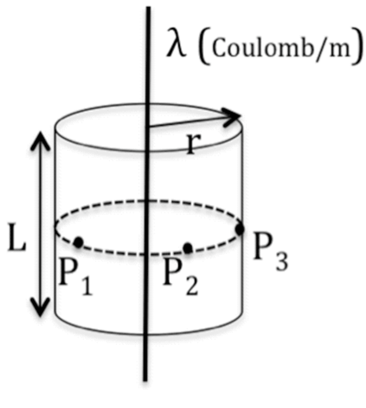


Figure 18.11:

Chapter 19

Magnetic Fields, Electric Currents and Ampere's Law

Moving charges make magnetic fields and magnetic fields effect moving charges.

19.1 Magnetism

Magnetism was known since ancient times as a property of some materials like Fe (iron), Ni (nickel) and Co (cobalt), ("*ferromagnetic*" materials, as we now call them), resulting in forces between magnetic objects. Magnetic objects have "*poles*" such that the magnetic force between two objects depends on the orientation of their poles. It was also known that the Earth has a magnetic field which orients compass needles in the North- South direction. The compass needle points to the magnetic poles of the Earth, which are not exactly at, but close to, the geographic North and South poles. Hence the poles of any magnet are designated as North and South poles, rather than (+) and (−) as we do for electric charges.

The first systematic empirical investigation of magnetism of the Earth and of magnetic materials came with the renaissance, with the work of William Gilbert in particular[1]. Electric and magnetic phenomena were thought to be unrelated until experiments showed that moving electric charges (*currents*) act as sources of magnetic field, and magnetic fields in turn effect moving electric charges (currents). With these experiments, performed at the end of the 18^{th} Century and in the early 19^{th} Century, it was realized that electric and magnetic phenomena are not independent, but they are coupled with each other. The full understanding of the interactions of charges, currents, electric and magnetic fields was one of the main triumphs of Physics in the 19^{th} Century, developed with the work of Ampere, Faraday and many others, and culminating in the synthesis of Electricity and Magnetism theory by Maxwell. The other main scientific achievements of the 19^{th} Century were the understanding of the Conservation of Energy and the Laws of Thermodynamics, in Physics, and Darwin's Theory of Evolution in Biology.

19.2 Currents Make Magnetic Fields: Ampere's Law

Moving electric charges, *currents*, set up magnetic fields[2]. Current, usually denoted with I, is the amount of charge moving across a cross section of the wire per unit time. The SI unit of current is the *Ampere*, abbreviated A. One Ampere is defined as a current of one

[1] see Gribbin, Chapter 3

[2] As you see in Experiment 2 of NS101

Coulomb of charge per second, $1\ A = 1\ C\ /\ s$. This unit is named after Ampere, who was the leading scientist in the investigation of the relation between currents and magnetic fields.

> The current going through any area, say an area inside some loop in air or in a plasma, is just the total amount of charge crossing that area per unit time. A related concept is the *current density*, usually denoted with the vector $\mathbf{j}$, is the amount of charge going through unit area per unit time. If there are n charge carriers per unit volume, each carrying charge q and moving with velocity $\mathbf{v}$, the current density $\mathbf{j}$ through a surface of unit area perpendicular to $\mathbf{v}$ is $\mathbf{j} = n\ q\ \mathbf{v}$ (see Problem 2). The total current through a surface element of area $\mathbf{dS}$ is simply $\mathbf{j} \cdot \mathbf{dS}$.

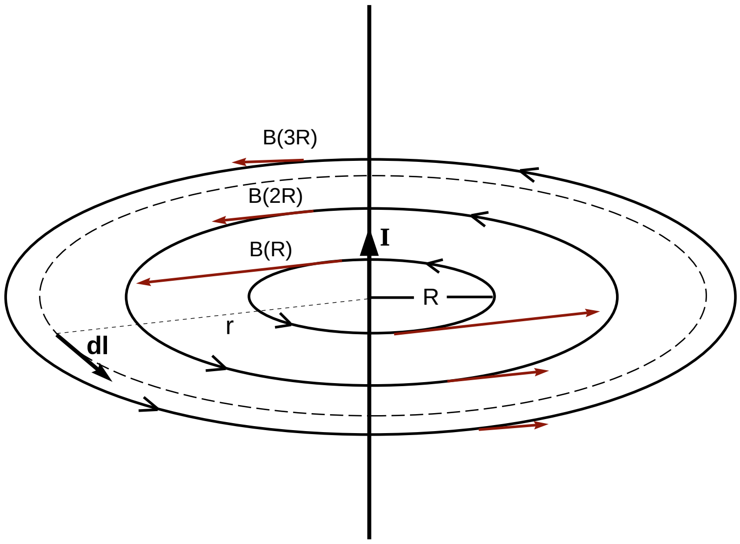


Figure 19.1:

In Figure 19.1 you see a *current* carrying wire. Current sets up a magnetic field. Experiments show the following properties of this magnetic field:

By symmetry, all points at the same distance from the wire have the same magnetic field[3].

1. The direction of the magnetic field, B, is tangential to the circles shown, the magnetic field circulates around the wire. Convention assigns the direction of the magnetic field as the direction from North to South for the magnetic field outside a simple bar magnet. Applying this definition to the present example, it turns out that the sense of circulation of the magnetic field with respect to the direction of the current obeys the right hand rule: If you point the thumb of your right hand in the direction of the current, the fingers of your right hand point in the direction of circulation of the

[3]Figure 19.1 is taken from the simulation "Ampere's Law" on the NS 101 SUCourse site. The compass in the simulation measures the magnetic field. Move the compass to see how the magnetic field changes. You can change the magnitude and direction of the current.

magnetic field. The magnetic field direction reverses when the current flows in the opposite direction.

2. The strength of the magnetic field decreases with distance r away from the wire in inverse proportion to distance, $B(r) \propto r^{-1}$.

3. The strength of the magnetic field is directly proportional to the magnitude of the current, $B(r) \propto I$.

Putting these observations together, we find that the strength of the magnetic field is given by

$$B(r) = \frac{\mu_0 I}{2\pi r}. \tag{19.1}$$

The direction of the magnetic field is given by the right hand rule with respect to the direction of the current. This is the simplest form of Ampere's Law, for the very symmetric situation of the "infinite" straight current carrying wire. The constant of proportionality is written with a factor 2π in the denominator in order to give later expressions a simpler form. The constant μ_0 is called "*the permeability of free space.*" The value of μ_o in the SI system of units is

$$\mu_0 = 4\pi \times 10^{-7}\ kg\ m\ C^{-2}.$$

Ampere's Law expresses the relationship between the magnetic field and the current which is the source of the magnetic field. As first established by the experiments of Ampere and others in the first half of the 19^{th} century, the most general statement of Ampere's Law, expressed in the *integral form*, is:

$$\oint \mathbf{B} \cdot \mathbf{dl} = \mu_0 I_{enc} \qquad [\textbf{Ampere's Law}] \tag{19.2}$$

Here **dl** denotes a step along a closed loop enclosing the current. At each point on the closed path, **dl** is a vector directed tangentially to the path with magnitude dl, which is the infinitesimal length of the step along the path at that point. $\mathbf{B} \cdot \mathbf{dl}$ is the component of the magnetic field tangential to the path, times the step dl. Integration on a closed loop is denoted by a little circle on the integral sign. Thus, Ampere's Law states that the integral of $\mathbf{B} \cdot \mathbf{dl}$ taken around a circle (or any other closed loop) around the current I will give the value of the current that passes through any surface enclosed by the loop. The closed loop integral adds up the component of **B** along the path (along the steps **dl**). A nonzero result of the integral means a net *circulation* of the magnetic field: the field lines may not necessarily follow the loop exactly, but there is a component of the field lines curving along and around the loop.

Let us use Equation 19.2 to find the magnetic field at a point at distance r from an "*infinite*" straight wire. The assumption of an infinite wire is a good approximation to a real situation, where the length L of the wire is much longer than the distance r of the point from the wire, $L \gg r$, and when the point is also far from the endpoints of the wire. For the infinite straight wire, all points on a circle in a plane perpendicular to the wire and centered at the wire are symmetric and equivalent points. To apply Ampere's Law in an easy and straightforward way, it is natural to choose a circle which is centered at the wire and has a radius r, indicated as the dashed curve in Figure 19.1. Experiment shows that the magnetic field is tangential to this circle. Due to symmetry, the magnitude of the magnetic field is the same at all points on the circle.

By using Equation 19.2, we can now check what the general expression gives in the simple case of the magnetic field at a distance r from an infinite straight wire. Since the magnitude of the field does not change along the circle centered at the wire, and the direction of the field is tangent to the circle at every point,

$$\oint \mathbf{B} \cdot \mathbf{dl} = \oint B(r)dl = B \oint dl = B\ 2\pi r. \tag{19.3}$$

The right hand side of Equation 19.2, I_{enc} is the total current enclosed by the *chosen* loop. The enclosed current in the loop is just the current through the wire, I. Thus we find that the magnitude of the magnetic field is

$$B(r) = \frac{\mu_0 I}{2\pi r}, \tag{19.4}$$

as formulated earlier.

Ampere's Law holds for any closed loop, but it is only useful and practical to apply in situations with symmetry, like the infinite straight wire. One can show that Ampere's Law is valid generally, on any closed contour enclosing the wire, from its validity on the circle centered on the wire, by distorting the circle into an arbitrary closed loop and showing that this change of contour does not effect the value of the integral of $\mathbf{B} \cdot \mathbf{dl}$ (see Problem 3).

19.3 * Contribution of Pieces of Current to the Magnetic Field: the Biot-Savart Law

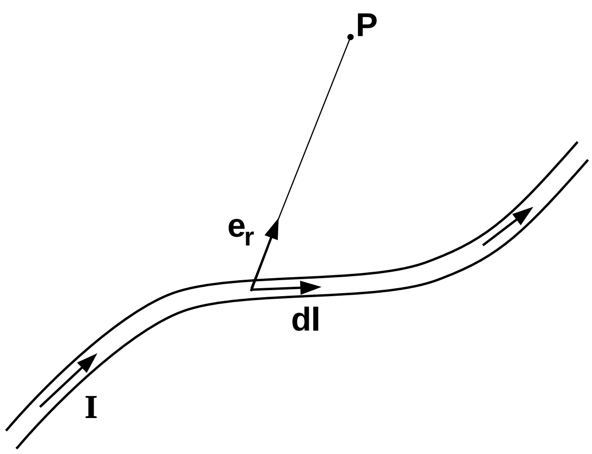


Figure 19.2:

Ampere's Law, like Gauss' Law, relates the *global* distribution of the field to its sources. In the case of Gauss' Law, an integral of the electric field, the *flux*, through a closed surface is related to the total charge enclosed. In the case of Ampere's Law, an integral of the magnetic field along a closed loop, the *circulation* is related to the total current enclosed by the loop. Either of these laws is particularly useful and practical in highly symmetric situations. If there is a distribution of currents that does not have any particular symmetry, one needs to find, at each point P in space, the contribution of each piece of current to the local magnetic field at P (see Figure 19.2). This is given by the Biot-Savart Law:

$$\mathbf{dB} = \frac{\mu_0}{4\pi}\frac{I\mathbf{dl}\times\hat{\mathbf{e}}_{\mathbf{r}}}{r^2} \tag{19.5}$$

established by experiments on asymmetric current distributions. Equation 19.5 gives the contribution $\mathbf{dB}$ to the local magnetic field due to each piece $I\mathbf{dl}$ of the source current. Thus the Biot-Savart Law is the analogue of Coulomb's Law, which gives the contribution to the local electric field from each individual point charge.

Since the magnetic field is proportional to (*linear* in) the current, the *Superposition Principle* applies:

> The total field at any point P is found by adding up (integrating) the contributions $\mathbf{dB}$ of all the bits of source current $I\mathbf{dl}$:

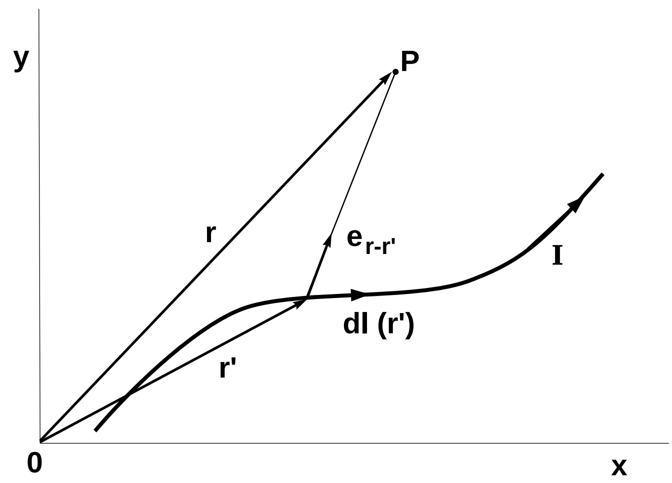


Figure 19.3:

$$\mathbf{B}(\mathbf{r}) = \int \mathbf{dB} = \frac{\mu_0}{4\pi}\int \frac{I(\mathbf{r}')\mathbf{dl}(\mathbf{r}')\times\hat{\mathbf{e}}_{\mathbf{r}-\mathbf{r}'}}{|\mathbf{r}-\mathbf{r}'|^2}. \tag{19.6}$$

Here $\mathbf{r}$ is the vector position of the point P where the magnetic field is measured, and $\mathbf{r}'$ denotes the variable points where the pieces $I(\mathbf{r}')\mathbf{dl}(\mathbf{r}')$ of the source currents are located, as seen in Figure 19.3. Integration over $\mathbf{r}'$ sums up the contributions of all bits of current located at all different positions $\mathbf{r}'$.

Note that the current $I(\mathbf{r}')$ passing through a point $\mathbf{r}'$ in the direction $\mathbf{dl}(\mathbf{r}')$ can be written in terms of the instantaneous charge $dq(\mathbf{r}')$ flowing through $\mathbf{r}'$.

$$I(\mathbf{r}')\mathbf{dl}(\mathbf{r}') = \frac{dq(\mathbf{r}')}{dt}\,\mathbf{dl}(\mathbf{r}') = \mathbf{v}(\mathbf{r}')dq(\mathbf{r}') \tag{19.7}$$

where $\mathbf{v}(\mathbf{r}')$ is the instantaneous velocity of the charges $dq(\mathbf{r}')$. So the Biot-Savart integral can also be written as:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{[\mathbf{v}(\mathbf{r}')\times\hat{\mathbf{e}}_{\mathbf{r}-\mathbf{r}'}]dq(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^2}. \tag{19.8}$$

19.4 Effect of a Magnetic Field on Electric Currents: the Lorentz Force

We have seen that moving electric charges and electric currents are the sources of magnetic fields. *Do magnetic fields have an effect on electric charges?*

Place an electric charge at rest in a magnetic field. Nothing happens; the charge remains at rest. *There are no forces on a charge at rest in a magnetic field.*

Now throw the charge, with velocity $\mathbf{v}$, into a region of space where there is a magnetic field. The charge is now accelerated. The acceleration is highest when the velocity $\mathbf{v}$ is perpendicular to the magnetic field. If the velocity is parallel to the magnetic field, there is no acceleration at all. These experiments give the result that there is a force

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B} \tag{19.9}$$

on a charge q moving with instantaneous velocity $\mathbf{v}$ through a point in space where the magnetic field has the value $\mathbf{B}$. If there is also an electric field $\mathbf{E}$, the total force is

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{19.10}$$

This is called the ***Lorentz Force***. The form of this force determines the SI unit of magnetic field, called *Tesla*, in honour of Nicola Tesla: 1 Tesla $\equiv$ 1 kg $\mathrm{s}^{-1}\mathrm{C}^{-1}$ (see Problem 14).

Force on a Current Carrying Wire

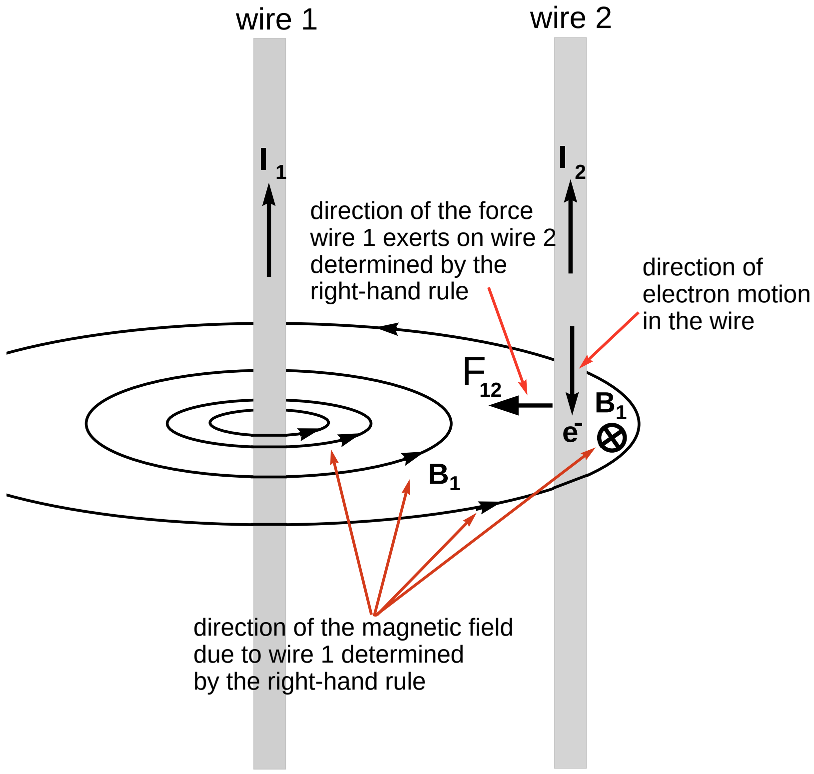
Consider two current carrying wires as in the figure below. We have the following information:

- Current is carried by charged particles, electrons in the case of a current carrying wire.
- Direction of current is defined to be opposite to the direction of electron motion (in other words: the direction of current is defined as the direction of motion of positive charge carriers).
- Due to Ampere's Law, a current carrying wire sets up a magnetic field in its surroundings.

Since the magnetic field set up by the wire exerts a force on charged particles, and since there is a bulk motion of charged particles in each current carrying wire, *the two wires should exert forces on each other.*

Continues on next page...

...Continued from previous page



In the figure above, the magnetic field set up by the first wire is shown. There is a bulk motion of electrons in the second wire towards the bottom of the figure. There is Lorentz force acting on these electrons by the magnetic field of wire 1. The direction of this force is pointing towards wire 1, that is, *wire 2 is attracted to wire 1.*

Similarly, the wire 2 sets up a magnetic field in its surroundings, and the bulk motion of electrons in wire 1 are affected by this magnetic field. The direction of this force is towards wire 2.

Therefore, *these two wires are attracted to each other...*

Also see problems 20.2 and 20.12.

CHAPTER 19 - PROBLEMS:

1. Sketch the magnetic field pattern of a bar magnet (a magnetic dipole). The magnetic field direction outside the bar magnet is from the North pole towards the South pole, by convention. This magnetic field pattern is the same as the pattern of electric field around an electric dipole in Chapter 16.

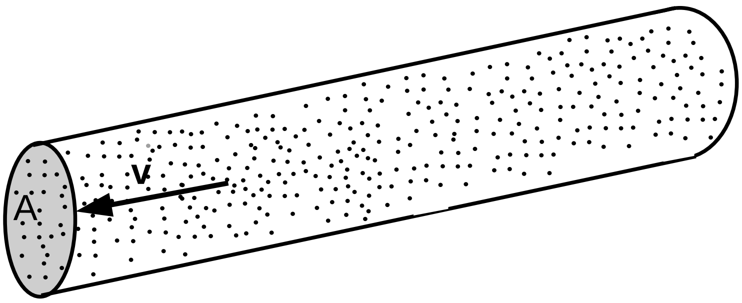


Figure 19.4:

2. Figure 19.4 shows a gas of $n = N/V$ charge carriers per unit volume, each with charge q, moving with average velocity $\mathbf{v}$. These particles could be electrons flowing in a metal wire or ions in a solution or plasma. Consider an area A perpendicular to $\mathbf{v}$ as seen in figure.
   (a) All charge carriers which are closer to A than the distance vt will reach and get through the area A within a time interval t. How many such carriers are there, and what is the total charge Q that will go through A in time t?
   (b) What is the current $I = Q/t$ through A ?
   (c) What is the magnitude of the current density $j = I/A$? $\mathbf{j}$ is in the same direction as the average particle velocity $\mathbf{v}$.
3. Consider a closed loop of arbitrary shape around an infinite straight wire carrying current I.
   (a) Can you make this loop of arbitrary shape from infinitesimal concentric circular arcs centered on the wire and radial segments?
   (b) Show that the circulation $\oint \mathbf{B} \cdot \mathbf{dl}$ on the loop of arbitrary shape is the same as the circulation on any circular loop centered on the wire.
4. Two very long parallel wires are separated by 10 cm, each carrying a current of 1 Ampere in the same direction. Make a figure and find the magnitude and direction of the magnetic field at 1 cm, 2 cm and 5 cm at either side from one of the wires.
5. * Take an infinite straight wire carrying current I. Use the Biot-Savart Law to calculate the magnetic field $\mathbf{B}$ at a point at distance R from the wire. You should obtain the same magnitude and direction for $\mathbf{B}$ that Ampere's Law gives in such a simple and direct fashion.

6. **The Solenoid:** Figure 19.5 (a) shows a long cylinder, of length L, around which a wire carrying current I is wound N times, with $n \equiv N/L$ windings per unit length. This arrangement is called a "*solenoid*". Figure 19.5 (b) shows a cross section of the solenoid, in a plane through its axis. Here $\otimes$ and $\odot$ denote current directions pointing into the page and out of the page, respectively.

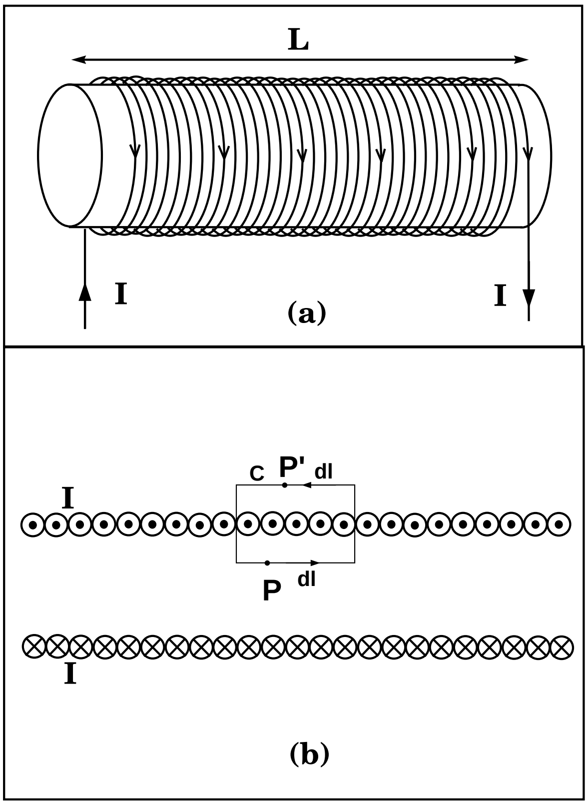


Figure 19.5:

   (a) What is the direction of the total magnetic field at a point like P inside the solenoid and far from the ends of the solenoid?

   (b) It is found experimentally that the magnetic field is zero at a point like P' outside the solenoid and far from the ends of the solenoid. Use Ampere's Law on the rectangular loop C shown in Figure 19.5, and symmetry arguments about the direction of the magnetic field, to find the magnitude of the magnetic field at a point P inside a very long ("infinite") solenoid. Does this result depend on where inside the solenoid the point P is?

7. Take the current configuration shown in Figure 19.6. Assume that all four wires, each being infinite and perpendicular to the page, carry the same current I. Use Ampere's Law to find the direction of the magnetic field at point P, the center of the square. Here $\otimes$ and $\odot$ denote current directions pointing into the page and out of the page, respectively.

8. Find the direction of the magnetic force for cases shown in Figure 19.7. Here $\otimes$ and $\odot$

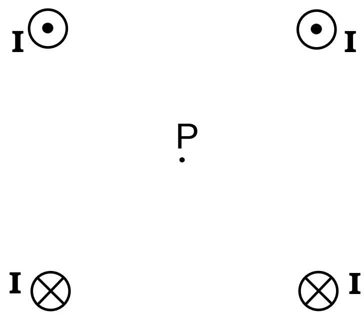


Figure 19.6:

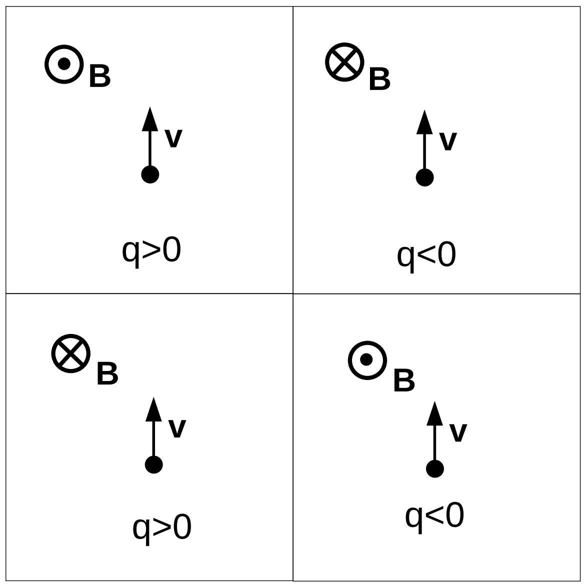


Figure 19.7:

denote magnetic field directions pointing into the page and out of the page (towards you), respectively.

9. A particle of charge $q > 0$ is moving in a region of uniform magnetic field and with velocity $\mathbf{v}$ perpendicular to $\mathbf{B}$ as shown in Figure 19.8.

   (a) Show the direction of the force.

   (b) The charge is observed to follow a circle. Why?

   (c) If the mass of the particle is m, find the radius r of the circle.

10. A particle of mass m and charge q has an initial velocity $\mathbf{v}$ going into a region with uniform magnetic field $\mathbf{B}$. The velocity $\mathbf{v}$ is neither parallel nor perpendicular to the magnetic field, but makes an angle θ with it. What is the trajectory of the charge?

11. **Force free motion:** A particle of mass m and charge q moves with constant velocity

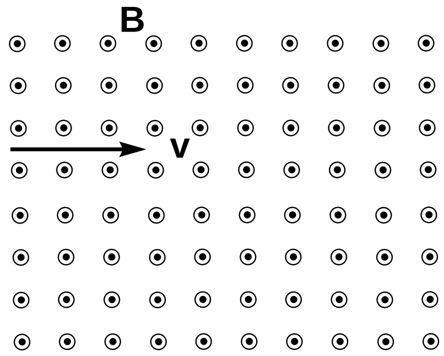


Figure 19.8:

v in a region where there is a uniform electric field **E** and a uniform magnetic field **B**. What is the relation between **E**, **v** and **B**?

12. A straight piece of wire of length L and cross sectional area A carrying current I is placed in a uniform magnetic field $\mathbf{B} = B_0\hat{\mathbf{j}}$. The current is carried by electrons, of charge $-e$, moving with average velocity $\langle \mathbf{v} \rangle = -v_0\hat{\mathbf{i}}$ (Figure 19.9). The number density (number per unit volume) of current carrying electrons is n.

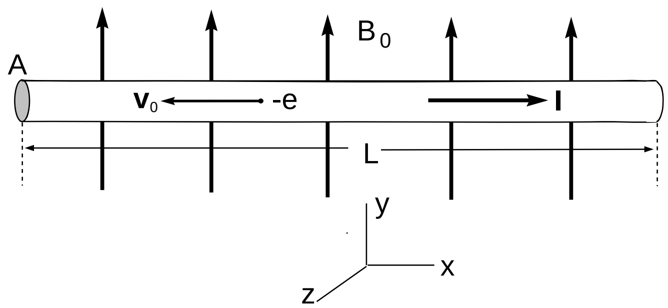


Figure 19.9:

(a) Express the current I in terms of the current density

$$\mathbf{j} = nev_0\hat{\mathbf{i}}$$

(b) Find the total Lorentz force on this piece of wire.

13. Two parallel infinite straight wires, with a separation D between them, each have charge λ $(C\ /\ m)$ per unit length. The charge carriers are moving with speed v along the wires. The direction of the flow of charges is the same in both wires.

(a) Make a picture of this arrangement.

(b) What is the current I on each of the wires?

(c) What are the magnitude and direction of the magnetic field **B** at the location of wire 2, due to the current in wire 1?

(d) What is the magnetic force $\mathbf{F}_{B,12}$, per unit length, that wire 1 exerts on wire 2?

(e) What are the magnitude and direction of the electric field $\mathbf{E}$ at the location of wire 2, due to the charge on wire 1?

(f) What is the electric force $\mathbf{F}_{E,12}$, per unit length, that wire 1 exerts on wire 2?

(g) What is the total Lorentz force, per unit length, that wire 1 exerts on wire 2?

(h) Can the total Lorentz force be zero? For what value of the carrier speed v?

14. The Lorentz Force, Equation 19.10, gives the dimensions and units of magnetic field $\mathbf{B}$ in terms of the basic dimensions mass, length, time and charge and in the corresponding SI units. Use the units and dimensions of $\mathbf{B}$ in the simple form of Ampere's Law, Equation 19.2 to obtain the dimensions and SI units of the proportionality constant μ_0.

Chapter 20

Cutting Electric and Magnetic Dipoles

Ever since the first experiments with electric charges, it has been well known that positive or negative charges can be separated from each other. By contrast, the two kinds of magnetic pole ("North" and "South") are never seen separated from each other.

> The charges in an electric dipole can be moved apart.
> The poles of a magnetic dipole cannot be moved apart.

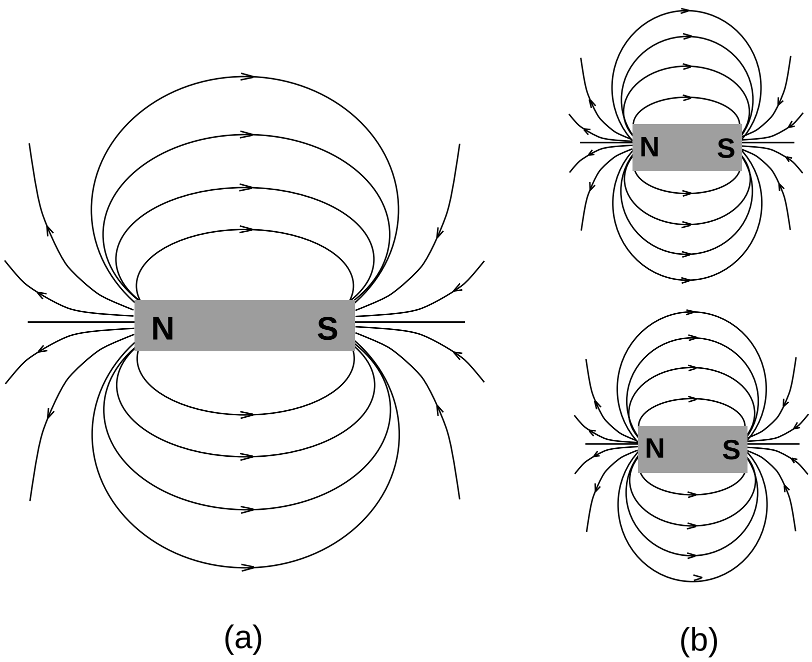


Figure 20.1:

There are no isolated magnetic North poles or isolated magnetic South poles in nature. No *magnetic monopoles* were found in any of the experimental searches so far. The simplest magnet is the bar magnet with North and South poles at its ends. This is a *magnetic dipole*. Any attempts to break the bar magnet into two result in two pieces which are themselves bar magnets with South and North poles, rather than a separate North monopole and a South monopole, as shown in Figure 20.1[1] If you cut a magnet, you obtain two new magnets, each with its own North and South poles. Magnetic fields are not set up by magnetic charges (monopoles). These do not exist in Nature.

Magnetic field lines of a magnetic dipole emanate from the North pole and converge onto the South pole. The pattern of this *dipole* magnetic field can be found by a simple experiment, by just spreading some iron dust around the bar magnet. The pattern is the same as the pattern of the electric dipole discussed in Section 16.2 (see Figure 20.1).

Electric dipoles, in contrast with magnetic dipoles, can be separated into their (+) and (−) charged components. As we saw in Chapter 16, an electric dipole is a pair of charges, one positive and the other negative, held apart from each other by some combination of forces overcoming their electrostatic attraction. Many electrically neutral molecules, including many biomolecules, have dipole moments that arise from the distribution of all their electrons. For example, in an ionic bond between two atoms, as in NaCl, there is extra electronic charge on one atom (Cl) kept apart from the positive charge arising from the loss of an electron by the other atom (Na). Another example is a dipole antenna which has a time dependent separation of charges driven by the signal (time dependent electric fields) received by the antenna. Dissociation of electric dipoles happens all the time in nature, for example, when you dissolve table salt, NaCl, in water. When an atom is ionized or an ionic molecule is dissociated, or when salt is dissolved in water, positive and negative charges are separated.

A bar magnet or a *solenoid* is a magnetic dipole. An isolated North or South pole of a magnet is never seen. As we have seen with the solenoid, magnetic fields are set up by currents of electric charges. This is also true for bar magnets (ferromagnets) which are made of atomic currents and elementary (fundamental, inseparable) *magnetic dipole moments* of electrons, called *spin*. *Nuclear Magnetic Resonance* (NMR) technology used in medicine is based on the magnetic dipoles (spins) of atomic nuclei. Since there is no evidence of isolated magnetic *monopoles*, the smallest and simplest magnetic source is a dipole. The simplest magnetic field pattern is the dipole field of the bar magnet or solenoid. A spherically symmetric monopole magnetic field pattern has never been observed.

Since magnetic fields are set up by electric currents, or by microscopic magnetic dipoles like the electron spin, and not by magnetic charges, one can see why cutting a magnetic dipole leads to two new dipoles. If you cut the solenoid in Figure 20.2(a) in the middle by cutting the current carrying wire, after a short while there is no current running in the now broken circuit, so the magnetic field is zero. To have a nonzero field with the two halves of the old solenoid, the cut ends of each piece of wire must be connected back to its other end through a circuit with a current source. But now there are two solenoids, each with its own north and south poles, as in Figure 20.2 (b). The statement that there are no magnetic charges (monopoles, isolated North and South poles) in nature means that Gauss' Law for magnetic field has zero on the right hand side:

$$\oint \mathbf{B} \cdot \mathbf{dS} = 0. \tag{20.1}$$

The total flux of magnetic field out of any closed surface is zero. This is one of Maxwell's equations (see Chapter 23). While it is simple to state, this equation provides an important

[1] This figure is taken from the simulation on NS 101 SUCourse under the title "What happens when you cut a magnet?".

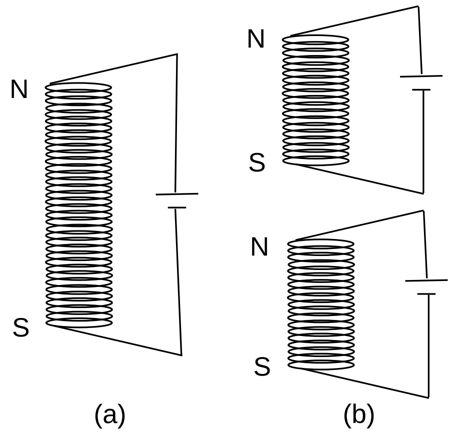


Figure 20.2:

constraint on the possible electromagnetic field configurations set up by a distribution of electric charges and currents. With electric charges and electric currents present, but their magnetic counterparts non-existing, there is no full symmetry between the way the electric field and the magnetic field appear in Maxwell's equations. In vacuum, however, Maxwell's Equations are fully symmetric with regard to electric and magnetic fields.

CHAPTER 20 - PROBLEMS:

1. A solenoid is cut in the middle, and the wires on each of the new half solenoids are connected to current sources. The two new solenoids are separated to a distance $D = 2.5\ L$ between their centers, where L is the length of each of the new solenoids.

   (a) Make a figure and sketch the magnetic field patterns for each of the new solenoids.

   (b) Sketch the pattern of the total magnetic field of the two half solenoids.

   (c) Sketch the magnetic field pattern of the initial single solenoid of length $2L$ and compare with the pattern in b).

2. Let us compare electric and magnetic monopoles as part of the corresponding dipoles and see what the flux over a closed surface is in each case.

   (a) Sketch the electric field pattern of an electric dipole (with one positive and one negative charges) in all space.

   (b) Draw a closed surface of any shape that includes one of the charges.

   (c) Look at the directions of the field line arrows crossing the surface. Is the total flux outward, inward, or both?

   (d) Sketch the magnetic field pattern of a magnetic dipole (with North and South poles) in all space, including between the poles.

   (e) Draw a closed surface of any shape that includes one of the poles.

   (f) Look at the directions of the field line arrows crossing the surface. Is the total flux outward, inward, or both?

   For an electric monopole (= single charge), the flux over a closed surface around the monopole is non-zero. However, for a magnetic monopole (= single pole), the flux over a closed surface around the monopole is in both ways canceling out each other, and the total flux turns out to be zero ($\oint \mathbf{B} \cdot \mathbf{dS} = 0$). This is because the magnetic "monopole" is not really an isolated monopole, it is always a part of a dipole.

Chapter 21

Changing Electric Fields Makes Circulating Magnetic Fields: the Displacement Current

Consider an infinite straight wire carrying current I. What happens if the wire is cut, leaving a gap where there is no current?

Suppose capacitor plates are placed at the gap as shown in Figure 21.1. In between the capacitor plates there is no current; let us do this in vacuum, so there is not even any matter to carry a current. Current is still arriving at the lower plate, charging it positively, and current is leaving the upper plate, charging it negatively.

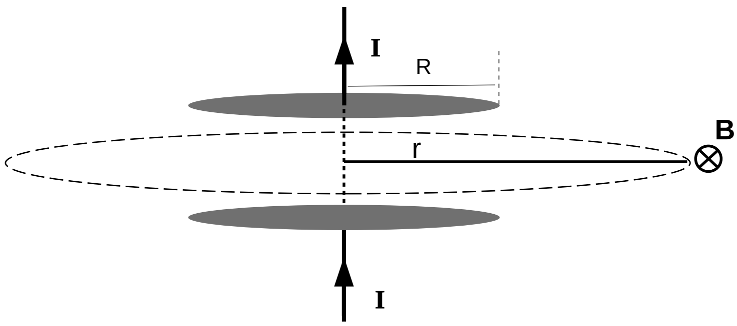


Figure 21.1:

Since there is current in the pieces of wire, magnetic fields circulating around the wire exist, and have the $1/r$ dependence on distance r from the axis given by Ampere's Law. But exactly the same magnetic fields are also measured in the space around the capacitor gap, where there are no currents! The magnetic field is found to be continuous in space as we move down parallel to the wire-capacitor system, from the regions around the current carrying wire into the regions where there is no current but only the capacitor gap. This is expected from the structure of the Maxwell equations, as Maxwell realized, before there was any experimental demonstration.

Experiments show that there is a circulating magnetic field around the capacitor plates[1]. What is then inside the loop of magnetic field lines? What is the source of the magnetic field? Well, between the capacitor plates there is an electric field. The magnitude of the

[1]The simulation in NS 101 SUCourse under "Ampere's Law" shows the magnetic field surrounding the plates

electric field is just proportional to the charge on the capacitor plates. In an arrangement like that shown in Figure 21.1, the presence of the current-carrying wires means the positive and negative charge on the two capacitor plates are changing all the time, with $I = dQ/dt$. This means there is a *time dependent* electric field in the empty space between the capacitor plates.

When there are time dependent electric fields, they set up a magnetic field circulating around the region even when there are no currents. The direction of the induced circulating magnetic field obeys the *right hand rule* around the direction in which the electric field is increasing. If you point the thumb of your right hand in the direction in which the electric field increases, the fingers of your right hand curve in the direction of circulation of the magnetic field. The induced magnetic field circulates in the *counterclockwise* direction around the direction of increase of the electric field. Ampere's Law is generalized in the form

$$\oint \mathbf{B} \cdot \mathbf{dl} = \mu_0 I + \mu_0 \epsilon_0 \int \frac{\partial \mathbf{E}}{\partial t} \cdot \mathbf{dS} \qquad [\mathbf{Ampere's\ Law, General}] \tag{21.1}$$

In SI units the coefficient in front of the new term contains the constant ϵ_0 of the Coulomb's and Gauss' Laws as well as the constant μ_0 of the first term in the Ampere's Law. Recall that the notation $\partial/\partial t$, called "*the partial derivative with respect to time*" simply means ignoring any dependence on other variables like x, y, z and calculating the rate of change with respect to time at constant x, y, z etc.

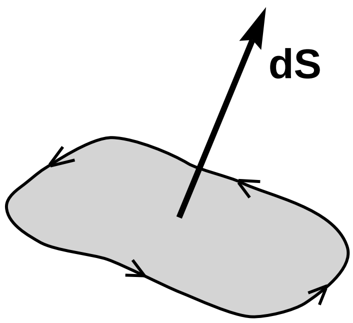


Figure 21.2:

What is the meaning of the integral in this new term? The integration takes place over the area enclosed by the closed loop of the left hand side of the Equation 21.1. The direction of the area elements **dS** are defined in association with the sense of circulation on the closed loop, in other words the orientation of the steps **dl** in tracing the loop. The sense of circulation is defined, by mathematical convention, according to the right-hand rule. If you place your right hand thumb along **dS**, the line elements **dl** on the loop point in the direction of the fingers of your right hand (see Figure 21.2). Thus **dl** must be oriented counterclockwise with respect to **dS**. With this definition of the *orientation* of **dl** with respect to **dS**, we can now state the physical relation between the electric and magnetic fields as derived from experiments:

> The magnetic field circulates around the direction of $\partial\mathbf{E}/\partial t$ according to the right hand rule.

The second term in Ampere's Law is the term added by Maxwell. This term links the electric and magnetic fields in vacuum to each other dynamically:

> If there is a *changing* electric flux, there will be a *circulating* magnetic field on closed loops encircling the region where electric flux is changing.

This effect is expressed in the last term in Equation 21.1. This term is analogous to *Faraday's Law of Induction* that is covered in Chapter 22. Only the role of the electric and magnetic fields are interchanged, and there is a minus sign in Faraday's Law, but a plus sign in the second term of Ampere's Law. Together, Ampere's and Faraday's Laws require that electric and magnetic fields can propagate as waves even in vacuum (and also in matter).

The quantity,

$$\epsilon_0 \int \frac{\partial \mathbf{E}}{\partial t} \cdot \mathbf{dS},$$

proportional to the time rate of change of the electric flux is called the "*Displacement Current*". The displacement current enters the full form of Ampere's Law in exactly the same way as the electric current. The displacement current is just as much a source of the circulating magnetic field as the electric current is.

CHAPTER 21 - PROBLEMS:

1. (a) Calculate the magnetic field at a distance $r < R$ away from the axis of the capacitor in Figure 21.1.

   (b) Calculate the magnetic field at a distance $r > R$ away from the axis of the capacitor in Figure 21.1.

   The area of the capacitor plates is $A = \pi R^2$. Take the electric field between the capacitor plates is uniform and equal to the value near the axis. Take the electric field to be zero outside the region between the plates.

2. Why don't we do this experiment with $\partial \mathbf{E}/\partial t$ in the lab?

   (a) What is the current needed to generate a magnetic field comparable to the Earth's magnetic field at a distance of 2 cm from the axis of the capacitor. The magnetic field of the Earth is $B_E \sim 4 \times 10^{-5}\ T$.

   (b) Calculate the voltage difference between the capacitor plates when the current that you found in part (a) charges the capacitor for 1 s.

3. A circular parallel plate capacitor of radius $R = 1\,\mathrm{cm}$ is connected to a time-dependent voltage, $V(t) = (10\,V)\sin(\omega t)$, with a frequency of $f = 100\,\mathrm{Hz}$. The capacitance of the capacitor is $2.0\,\mu\mathrm{F}$.

   (a) Find the charge, $q(t)$, on the capacitor.

(b) Find the displacement current, I_d, inside of the capacitor.

(c) Find the electric field, $E(t)$, inside of the capacitor.

(d) Find the magnetic field, $B(t)$, outside of the capacitor at a distance r from the capacitor's axis.

Chapter 22

Faraday's Law of Induction

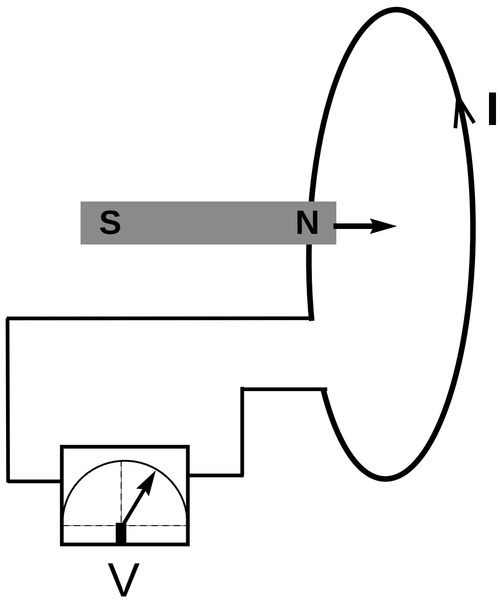


Figure 22.1:

In Figure 22.1 there is a circuit – a loop of wire – with no source of voltage attached to it. So there is no electric field in the wires, nothing to accelerate electrons[1].

And you have a magnet. When the magnet is at rest with respect to the loop of wire, the magnetic field configuration is static. There is no electric field, there is a magnetic field **B**.

Now move the magnet towards the wire. Suddenly the voltmeter shows that there is a voltage, the electric field in the wires is no longer zero. As the magnet moves *faster* towards the wire loop, the electric field increases. If the magnet is very close to the loop but it is at rest, the electric field is zero. There is an *induced* electric field only when the magnet moves and the magnetic fields through the loop are varying[2]. If you move the magnet backwards the electric fields (voltmeter readings) are in the opposite direction. Doing this kind of experiment with many different situations with changing magnetic fields, Faraday found that

[1]This picture is taken from the simulation in NS 101 SUCourse under "Faraday's Law"

[2]In the SUCourse simulation you can observe the voltage readings when the magnet is moving through the loop. If you move the same magnet at a faster velocity (click v2) the electric fields registered are larger.

when the flux of the magnetic field through a loop is changing, there is an electric field with a net circulation along the loop. The magnitude of the electric field is proportional to the rate of change of the magnetic flux through the loop. A very strong magnetic field with no changes does not give an electric field. The formulation of Faraday's law, summarizing all the experiments, is:

$$\oint \mathbf{E} \cdot \mathbf{dl} = -\frac{d}{dt} \int \mathbf{B} \cdot \mathbf{dS} \qquad [\mathbf{Faraday's\ Law}] \tag{22.1}$$

The constant of proportionality between $\oint \mathbf{E} \cdot \mathbf{dl}$ and $-\frac{d}{dt} \int \mathbf{B} \cdot \mathbf{dS}$ has the value 1 in SI units.

Faraday's Law states that:

> The circulation of the electric field along any closed loop is equal to *minus* a constant times the rate of change of the magnetic flux through that loop.

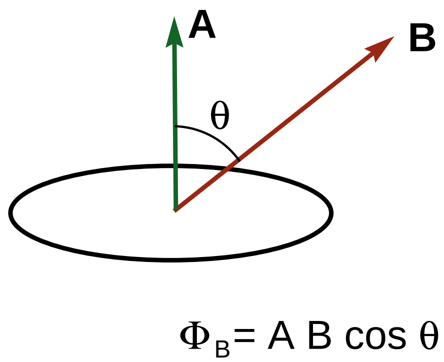


Figure 22.2:

What is the meaning of the minus sign on the right hand side of Equation 22.1?

In connection with the displacement current term in a similar expression, the generalized Ampere's Law (Equation 21.1) of Chapter 21, we discussed the convention for directions of **dl** and **dS**. The mathematical convention assigns the orientation of **dl** around a closed loop by the right hand rule. If you place your right hand thumb along **dS** the line elements **dl** on the loop point in the direction of your fingers, thus counterclockwise with respect to the area elements **dS**. The minus sign in Equation 22.1 expresses that if the magnetic flux ($\mathbf{B} \cdot \mathbf{dS}$) through the surface in the direction of **dS** is increasing, the circulating induced electric field would be oriented in the *clockwise* direction around the loop. The induced **E** is in the opposite direction to the orientation of **dl** on the loop. This induces a current also running in the clockwise direction around **dS**, if there are charges around available to be accelerated and set into motion by the electric field. There will be charges when we are not in vacuum, in particular when our loop is a physical piece of wire, or if it runs through some plasma.

For Figure 22.2, the magnetic flux ($\int \mathbf{B} \cdot \mathbf{dS} = \Phi_B$) is in the direction of the area element **A** (in the direction of **dS**):

$$\Phi_B = AB\cos\theta.$$

Therefore, an increase in the magnetic flux through a loop could be caused by an increase in the magnitude of the magnetic field, the area of the loop, or the increase in $\cos\theta$.

Question: What is the direction of the *secondary* magnetic field that this induced current sets up according to Ampere's Law?

Answer: For an increase of magnetic flux in one direction, the secondary magnetic fields are generated such that their flux through the loop is in the opposite direction.
Thus, a change in magnetic flux induces electric fields, which produce currents, which produce secondary magnetic fields in a direction opposite to the direction of the initial change in flux, acting to counteract the initial change! This is observed in the experiments and expressed through the minus sign in Faraday's Law. The minus sign, the magnetic-field-restoring character of the induced electric fields, is called ***Lenz' Law***. Lenz' Law is not really a separate law, it is just the minus sign aspect of Faraday's Law.

If there is a *changing magnetic flux* there will be a *circulating electric field* on closed loops encircling the region where magnetic flux is changing. Going around such a loop and coming back to the same point, there is a net change in voltage (called an *induced* voltage or "induced electromotive force", e.m.f.). This means a change in the potential energy of a charge moving around the loop even though there is no battery or other agent doing work on the charge!

Faraday's Law of Induction is analogous to the second term of Ampere's Law, the Displacement Current term. The role of the electric and magnetic fields are interchanged, and there is a minus sign in Faraday's Law that does not appear in the second term of Ampere's Law. The minus sign means there is a restoring character to the interaction between the electric and magnetic fields. The restoring effect imparts an oscillatory character to the coupled dynamics of electric and magnetic fields. Together, Ampere's and Faraday's Laws require that electric and magnetic fields can propagate as waves, even in vacuum (and also in matter).

CHAPTER 22 - PROBLEMS:

1. The induced voltage on a solenoid: Consider the solenoid, studied in Problem 19.6. Using Ampere's Law, the magnetic field inside an infinite solenoid with n windings per unit length, carrying current I was found to be $B = \mu_0 n I$. The approximation of an infinite solenoid is a good approximation far from the edges of a real finite solenoid of length l with N windings so that $n = N/l$. The cross sectional area of the solenoid is A, so the magnetic flux Φ_B inside the solenoid is $\Phi_B = \mu_0 N I A/l$.
   (a) If the current is changing with time, with rate of change dI/dt, what is the induced voltage around one turn of the wire ?
   (b) What is the total induced voltage around the N windings making up the solenoid?

(c) According to Faraday's law the induced voltage around a loop is linearly dependent on the rate of change of the flux, $d\Phi_B/dt$ and through Ampere's Law, on the rate of change of the current, dI/dt. Thus the total induced voltage on any time dependent current carrying system has the form $V = -L\ dI/dt$, where the proportionality constant L, called the "*inductance*" is a property of the particular system. What is the inductance of the solenoid?

(d) What is the dimension and SI unit of inductance in terms of C, m, kg, and s? This unit is called the "*Henry*" for the American physicist Joseph Henry.

2. A solenoid of $n = 1000\ turns/m$ and cross sectional area of 1 cm^2 carries an alternating current $I(t) = I_0\ \sin(\omega t)$ where the current amplitude $I_0 = 2\ A$ and frequency $f = 50\ Hz$. What is the inductance L and the induced voltage $V(t)$?

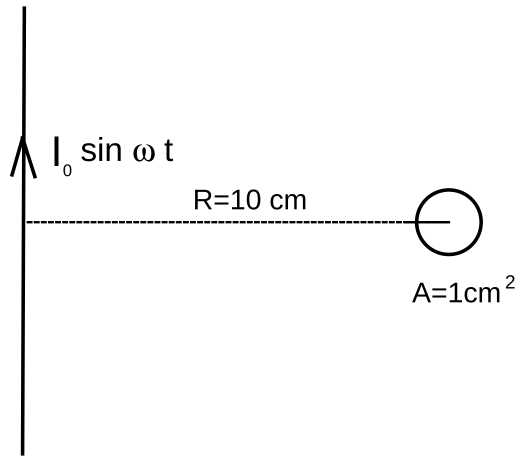


Figure 22.3:

3. An infinite straight wire is carrying an alternating current (AC) $I(t) = I_0 \sin(\omega t)$ where the current amplitude $I_0 = 5\ A$ and frequency $f = 50 Hz$ (See Figure 22.3). At a distance of 0.1 m from the straight wire there is a circular wire loop encompassing an area of 1 cm^2. The circular loop is in a plane that contains the infinite straight wire. Take the value of the magnetic field through the loop to be the value at its center, at 0.1 m from the straight wire. What is the induced voltage on the circular loop?

4. **The *Toroid*:** The circular loop in the previous problem is extended to 10 000 circular windings of the wire on the surface of a *torus* of cross sectional area 1 cm^2 at 1 m from the infinite straight wire. Such a winding is called a *toroid*. What is the induced voltage on this toroid?

5. The infinite straight wire in Problem 3 carries a *constant* (DC) current of 2 A. The same circular loop of wire is now rotated uniformly around an axis parallel to the infinite straight wire, with a period $P = 0.02\ s$ - a frequency $f = 50\ Hz$. Take the value of the magnetic field through the loop to be the value at its center, at 1 m from the straight wire. What is the induced voltage on the circular loop?

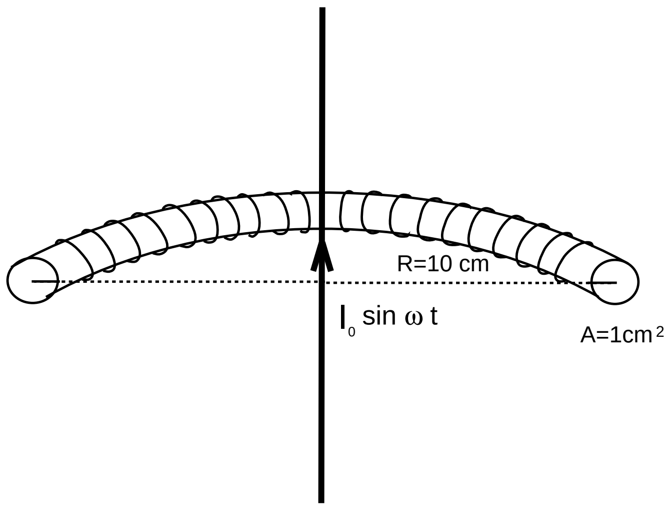


Figure 22.4:

Chapter 23

Waves from Maxwell's Equations

We have now seen all four ***Maxwell's equations*** of Electricity and Magnetism:

MAXWELL'S EQUATIONS

$$\oint \mathbf{E}\cdot\mathbf{dS} = \frac{Q_{\rm in}}{\epsilon_0} \qquad [\mathbf{Gauss'\ Law}]$$

$$\oint \mathbf{B}\cdot\mathbf{dS} = 0$$

$$\oint \mathbf{E}\cdot\mathbf{dl} = -\frac{d}{dt}\int \mathbf{B}\cdot\mathbf{dS} \qquad [\mathbf{Faraday's\ Law}]$$

$$\oint \mathbf{B}\cdot\mathbf{dl} = \mu_0 I + \mu_0\epsilon_0 \int \frac{\partial \mathbf{E}}{\partial t}\cdot\mathbf{dS} \quad [\mathbf{Ampere's\ Law, General}]$$

In this Chapter we will study the electric and magnetic fields between the "half-infinite" parallel plates shown in Figure 23.1 and express Ampere's Law and Faraday's Laws for these fields. Next we will see how wave equations can be obtained from them[1]. Let us first use the symmetry of the half-infinite parallel plate system. A little into the plates from the edge at $x = 0$, where the distance x from the edge of the plates is larger than the separation between the plates, the electric field will be in the z direction perpendicular to the plates carrying the charges. The charges on the plates vary with time because of the variable voltage difference imposed at $x = 0$. The electric field in the z direction is time dependent, with dE/dt also in the z direction. As the charges on the plates vary with time, time dependent currents will be going down the plates into the system in the x direction from the energy source at $x = 0$. The dE/dt in the z direction and the currents flowing in the x direction will act as sources for magnetic field. By the symmetry of the parallel plates the magnetic fields must be in the y direction: xy planes perpendicular to dE/dt or yz planes perpendicular to $i(t)$ both contain the y direction. In other words B should be circulating around dE/dt, which is in the z direction, and around $i(t)$ which is in the x direction. In the symmetry of the plane parallel plates this leaves the y direction for the B field. All points on a constant y line are physically equivalent. The symmetry leaves no possibility for curved magnetic field lines or magnetic fields to point in z or x directions.

[1] This is a mathematically more advanced chapter. Its physics content is the same as what you see in the "Electromagnetic Waves" simulation on NS 101 SUCourse. Do not worry if you do not follow the mathematics. The figures in this chapter are replicated from this simulation, they are snapshots of different frames of the simulation

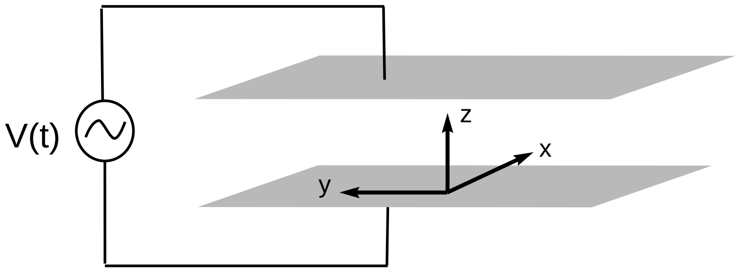


Figure 23.1:

23.1 Application of Ampere's Law

Consider the generalized Ampere's Law, including Maxwell's displacement current term (Equation 21.1)

$$\oint \mathbf{B} \cdot \mathbf{dl} = \mu_0 I + \mu_0 \epsilon_0 \int \left(\frac{\partial \mathbf{E}}{\partial t} \right) \cdot \mathbf{dS} \tag{23.1}$$

Since there are no charges or currents in the gap between the plates, the first term in Ampere's law does not contribute, so

$$\oint \mathbf{B} \cdot \mathbf{dl} = \mu_0 \epsilon_0 \int \left(\frac{\partial \mathbf{E}}{\partial t} \right) \cdot \mathbf{dS} \tag{23.2}$$

where the integral on the left hand side is to be evaluated around a loop and the integral on the right hand side is to be taken over any area surrounded by the loop. In the displacement current term we use $\partial \mathbf{E}/\partial t$, the partial derivative of $\mathbf{E}$ with respect to t.

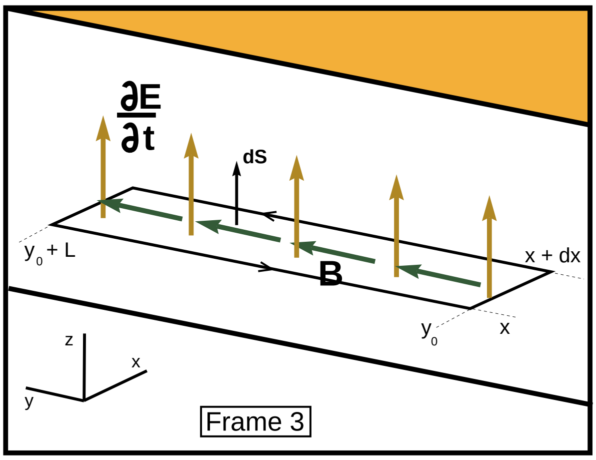


Figure 23.2:

We use the symmetry of the half-infinite parallel plates to express Ampere's Law in a particularly simple form. In Figure 23.2[2] the rectangular loop C extends from $y = y_0$ to

[2]Figure 23.2 is a snapshot of Frame 3 from the simulation in NS 101 SUCourse under Electromagnetic Waves

$y = y_0 + L$ between x and $x + dx$ on the xy-plane. Let us choose the direction of the area element $\mathbf{dS}$ to be in the $+z$ direction. Then the orientation of the steps $\mathbf{dl}$ in tracing the rectangular loop must be right handed, counter-clockwise as shown with arrows in Figure 23.2. On the short sides of the rectangle, $\mathbf{B} = \hat{\mathbf{j}}\ B_y$ is perpendicular to $\mathbf{dl} = \hat{\mathbf{i}}\ dx$ or $\mathbf{dl} = -\hat{\mathbf{i}}\ dx$, so $\mathbf{B} \cdot \mathbf{dl} = 0$ on the short sides.

Therefore, the only contribution comes from the long sides of the rectangle. On the long side of the rectangle at $x + dx$, $\mathbf{B} = B_y(x + dx)\ \hat{\mathbf{j}}$ is in the direction of $\mathbf{dl}$ tracing the loop, while on the long side at x, $\mathbf{B} = B_y(x)\ \hat{\mathbf{j}}$ is directed opposite to the direction of the steps $\mathbf{dl}$ tracing the loop.

Thus, the left hand side of Ampere's Law (23.2) is

$$\oint \mathbf{B} \cdot \mathbf{dl} = B_y(x + dx, t)L - B_y(x, t)L = L \left(\frac{\partial B_y}{\partial x}\right) dx \tag{23.3}$$

In the last step, we have used the definition of partial derivative:

$$\frac{\partial B_y}{\partial x} = \frac{B_y(x + dx, t) - B_y(x, t)}{dx}$$

On the right hand side of Ampere's Law (23.2), since $\mathbf{dS}$ is chosen to be in the $+z$ direction, the displacement current term is

$$\frac{\partial \mathbf{E}}{\partial t} \cdot \mathbf{dS} = \frac{\partial E_z}{\partial t}\ dS$$

The loop area $S = Ldx$ is infinitesimally small, therefore $\partial E_z/\partial t$ can be treated as constant over the loop area and can be taken outside the integral:

$$\mu_0\epsilon_0 \int \left(\frac{\partial \mathbf{E}}{\partial t}\right) \cdot \mathbf{dS} = \mu_0\epsilon_0 \left(\frac{\partial E_z}{\partial t}\right) \int dS = \mu_0\epsilon_0 \left(\frac{\partial E_z}{\partial t}\right) Ldx \tag{23.4}$$

Substituting what we have found for the left and right hand sides (namely (23.3) and (23.4)) in Ampere's Law (23.2), we obtain:

$$L \left(\frac{\partial B_y}{\partial x}\right) dx = \mu_0\epsilon_0 \left(\frac{\partial E_z}{\partial t}\right) Ldx \tag{23.5}$$

Eliminating L and dx on both sides, one gets Ampere's Law for this problem, expressed as a "differential equation":

$$\frac{\partial B_y}{\partial x} = \mu_0\epsilon_0 \frac{\partial E_z}{\partial t} \tag{23.6}$$

23.2 Application of Faraday's Law

Now, let us consider the integral form of Faraday's Law

$$\oint \mathbf{E} \cdot \mathbf{dl} = -\frac{d}{dt} \int \mathbf{B} \cdot \mathbf{dS} \tag{23.7}$$

where the integral on the left hand side is to be evaluated around a loop and the integral on the right hand side is to be taken over the area of the same loop.

Let us consider the imaginary rectangular loop C' in Fig.23.3[3]. The loop is in the

[3]Figure 23.3 is a snapshot of Frame 6 from the simulation in NS 101 SUCourse under Electromagnetic Waves

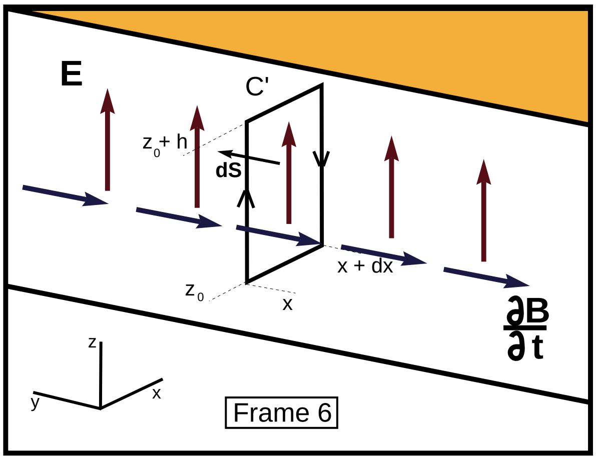


Figure 23.3:

xz-plane, extending from $z = z_0$ to $z = z_0 + h$, and from x to $x + dx$ between our parallel plates. The area of this loop is $dS = h\ dx$. On the short sides of the rectangle $\mathbf{E} = \hat{\mathbf{k}}\ E_z$ is perpendicular to $\mathbf{dl} = \hat{\mathbf{i}}\ dx$ or $\mathbf{dl} = -\hat{\mathbf{i}}\ dx$. So $\mathbf{E} \cdot \mathbf{dl}$ is zero on the short sides. The only contribution to the line integral of $\mathbf{E} \cdot \mathbf{dl}$ (the left hand side of Faraday's law) comes from the long (vertical) sides of the rectangular loop, which have length h. Let us take $\mathbf{dS}$ on the right hand side of equation (23.7) to be in the $+y$ direction, the rectangular loop C' on the left hand side must be traced in the direction shown in the Figure, right handed with respect to $\mathbf{dS}$.

$$\oint \mathbf{E} \cdot \mathbf{dl} = E_z(x,t)h - E_z(x+dx,t)h = -h\left(\frac{\partial E_z}{\partial x}\right)dx \tag{23.8}$$

The right hand side of Faraday's Equation contains the time derivative of the magnetic flux through the loop area in the $+y$ direction of our chosen $\mathbf{dS}$. As the loop area $S = h\ dx$ is infinitesimally small, B_y can be regarded as constant over it and can be taken outside the integral. We have

$$-\frac{d}{dt}\int \mathbf{B} \cdot \mathbf{dS} = -\frac{d}{dt}B_y\int dS = -\left(\frac{\partial B_y}{\partial t}\right)S = -\left(\frac{\partial B_y}{\partial t}\right)hdx \tag{23.9}$$

Substituting what we have found for the left and right hand sides (namely Eqns. (23.8) and (23.9)) in Faraday's Law (23.7), we obtain:

$$-h\left(\frac{\partial E_z}{\partial x}\right)dx = -\left(\frac{\partial B_y}{\partial t}\right)hdx \tag{23.10}$$

Eliminating h and dx on both sides, one gets Faraday's Law, expressed as a differential equation:

$$\frac{\partial E_z}{\partial x} = \frac{\partial B_y}{\partial t} \tag{23.11}$$

23.3 Obtaining the Wave Equation

Let us now put the differential forms of Ampere's and Faraday's Laws together:

$$\frac{\partial B_y}{\partial x} = \mu_0\epsilon_0\frac{\partial E_z}{\partial t} \tag{23.12}$$

$$\frac{\partial E_z}{\partial x} = \frac{\partial B_y}{\partial t} \tag{23.13}$$

Now, let us take the partial time derivative of both sides of Ampere's Law (23.12) and the partial x derivative of Faraday's Law (23.13), so the equations become:

$$\frac{\partial^2 B_y}{\partial t\partial x} = \mu_0\epsilon_0\frac{\partial^2 E_z}{\partial t^2} \tag{23.14}$$

and

$$\frac{\partial^2 B_y}{\partial x\partial t} = \frac{\partial^2 E_z}{\partial x^2} \tag{23.15}$$

The left hand sides of the last two equations are identical.
This means that their right hand sides are equal to each other. In other words:

$$\frac{\partial^2 E_z}{\partial x^2} = \mu_0\epsilon_0\frac{\partial^2 E_z}{\partial t^2} \tag{23.16}$$

This is the ***wave equation*** for the electric field. The solution of this equation is:

$$E_z = E_0\cos(kx - \omega t + \phi) \tag{23.17}$$

The constants appearing in this solution reflect important wave properties. The constant ϕ, called the ***phase***, depends on the choice of the origin of time and the conditions on the boundary of the system. In our example, we can choose these such that $\phi = 0$, if the electric field applied to the plates at $x = 0$ is

$$E_z(x = 0, t) = E_0\cos(\omega t) \tag{23.18}$$

Then the solution is

$$E_z = E_0\cos(kx - \omega t) \tag{23.19}$$

To study the x dependence of the electric field, take a snapshot of the solution at any fixed moment t in time. The x dependence has the shape of a cosine wave. The electric field has the same values repeated at any x and at a point $x + \lambda$ such that $k\lambda = 2\pi$. λ is called the ***wavelength*** and k is the ***wave number*** ($k \equiv 2\pi/\lambda$). Now consider the solution for the electric field at a fixed point x. As time goes on the value of the electric field oscillates, as a cosine function. The value of the electric field at time t will be repeated at a later time $t+T$ such that $\omega T = 2\pi$. T is called the ***period*** and ω is the ***angular frequency*** ($\omega \equiv 2\pi/T$). At any particular x, the electric field is going through a simple harmonic oscillation.

23.4 Electromagnetic waves propagate with the speed of light:

Now let us consider the ***propagation*** of this wave. Writing the electric field solution as

$$E_z = E_0\cos[k(x - vt)], \qquad v = \frac{\omega}{k}, \tag{23.20}$$

we see that the wave has the same value at the point $x = x_0 + vt$ at time t as it had at point x_0 , at time $t = 0$. *The wave is traveling in the positive x direction with velocity* $v = \omega/k$. Note that

$$v = \frac{\omega}{k} = \left(\frac{2\pi}{T}\right)\left(\frac{\lambda}{2\pi}\right) = \frac{\lambda}{T} \tag{23.21}$$

If the wave solution (23.17) is substituted in the wave equation (23.16), the two sides of the equation are calculated as follows:

$$\frac{\partial^2 E_z}{\partial x^2} = -k^2 E_0 \cos(kx - \omega t + \phi) \tag{23.22}$$

$$\mu_0\epsilon_0 \frac{\partial^2 E_z}{\partial t^2} = -\mu_0\epsilon_0\omega^2 E_0 \cos(kx - \omega t + \phi). \tag{23.23}$$

In order to satisfy Equation (23.16), we must have

$$\mu_0\epsilon_0\omega^2 = k^2.$$

By Equation (23.21) this implies

$$v^2 = \frac{1}{\mu_0\epsilon_0} \tag{23.24}$$

Using $\mu_0 = 4\pi\times10^{-7} T\ m\ A^{-1} = 4\pi\times10^{-7} kg\ m\ C^{-2}$ and $\epsilon_0 = 8.854\times10^{-12} C^2\ N^{-1}\ m^{-2}$, v is found to be numerically equal to the speed of light $c = 3.00 \times 10^8 m/s$.

$$c = \frac{1}{\sqrt{\mu_0\epsilon_0}} \qquad [\textbf{Speed of Light}] \tag{23.25}$$

In a similar way the magnetic field B_y also satisfies the wave equation. To see this, divide both sides of Ampere's law (23.12) by $\mu_0\epsilon_0$ and take its partial x derivative. Also take the partial time derivative of Faraday's Law (23.13). Comparing the two equations we find:

$$\frac{\partial^2 B_y}{\partial x^2} = \mu_0\epsilon_0 \frac{\partial^2 B_y}{\partial t^2} \tag{23.26}$$

The solution of this equation is:

$$B_y = B_0 \cos(kx - \omega t + \phi) \tag{23.27}$$

where k and ω have the same physical meaning as in the case of the wave equation for the electric field.

> The electrical and magnetic fields both propagate as waves in vacuum. The directions of the electrical field, the magnetic field and the direction of propagation of the wave are mutually perpendicular.

The electrical and magnetic fields are related to each other through Ampere's and Faraday's Laws, equations (23.1) and (23.7). The values of the constants E_0, B_0 and ϕ are fixed by the boundary conditions. Using either Equation 23.12 or 23.13 with the solutions (E_z and B_y), it is found that

$$B_0 = -E_0/c. \tag{23.28}$$

The boundary condition $E_z(x=0,t) = E_0\cos(\omega t)$, the electrical field supplied at the edge of parallel plates by the external AC voltage source, fixes $\phi = 0$ in both the electric and the magnetic field solutions, through Equations 23.12 and 23.13.

We have thus found out that light is an electromagnetic wave. Light is not the only kind of electromagnetic wave. Maxwell's equations and the wave equation are not restricted to any particular wavelengths or frequencies. Electromagnetic waves of all frequencies and wavelengths will propagate in vacuum provided that $\omega/k = c$ (or $\lambda\nu = $ c), since there is no special scale of length or time in Maxwell's Equations. All sorts of electromagnetic waves exist and they all travel at the velocity of light c in vacuum. The full range of wavelengths and frequencies, the "***spectrum***" of electromagnetic radiation, is very wide, extending from radio waves at long wavelengths, of the order of kilometers, to *gamma rays*, at frequencies of order $10^{23} Hz$ or higher and corresponding wavelengths of the order of or less than 10^{-15} m.

When Maxwell showed that electromagnetic wave propagation at the velocity of light followed from the Maxwell Equations, light was the only known form of electromagnetic radiation. Hertz did the first experiments on the lines of the Maxwell Equations to generate and detect radio waves, which later inventors like Marconi developed into the technology of electromagnetic communications. With the late 19^{th} Century through early 20^{th} Century development of atomic and nuclear physics, came the discovery of X- and gamma rays and the recognition that these were very high frequency electromagnetic radiation.

The frequencies of electromagnetic waves are commensurate with the oscillation frequencies of charge and current systems (time dependent *circuits* or *antennae* in the most general sense) that produce them, including light sources, atoms, radioactive nuclei emitting of gamma rays, and so on. The wavelengths are likewise determined by the sizes (*near zones*) of the source systems. The propagation of electromagnetic waves is determined by the interaction of the electromagnetic fields with the matter they encounter. *Resonant* frequencies that coincide with the oscillation frequencies of the material medium will be *absorbed* or *scattered*, which means the fields in the wave will efficiently accelerate charges in the medium, and lose their energy. The charges may then re-emit new electromagnetic waves in other directions and possibly at other frequencies. As a result, electromagnetic waves of certain characteristic frequencies will not pass through the medium. The medium is *transparent*to electromagnetic waves of some frequencies and *opaque* to others. The detection and use of electromagnetic waves also requires *receivers* designed to resonate with the frequencies of interest; sometimes with a *tuner* that can adjust the receiving system parameters to respond to a frequency of choice.

Stars and galaxies emit electromagnetic radiation in all bands from radio to gamma rays. The longest wavelength radio waves do not leave the emitting regions because they are absorbed by even the lowest density of charged particles (*plasma*). The highest frequency gamma rays likewise do not propagate out of the emitting regions because the gamma-ray *photons* have such high energies that they immediately decay – creating other high energy particles. Electromagnetic waves in the wide spectrum in between the longest wavelength radio waves and the highest frequency gamma waves are emitted in various processes, in stars, galaxies, gas clouds, the interstellar medium etc.

Our Sun, like most stars, radiates its energy predominantly in the *optical band*, that is, as *light*. The Earth's atmosphere is opaque to most bands of electromagnetic radiation. The only *windows*, frequency bands in which the atmosphere is transparent, are the optical band, and some parts of the radio and microwave bands.

The 3.5×10^9 years of evolution has produced species that are naturally sensitive to the electromagnetic radiation that the Earth's atmosphere allows, namely light, and vulnerable to radiation to which the atmosphere is opaque, like X-rays and ultraviolet. Our eyes are sensitive to electromagnetic radiation with wavelengths between ~3500 Angstroms (350 nanometers)[4], which we sense as violet, and ~6500 Angstroms (650 nanometers), which we sense as red.

When electromagnetic waves propagate in a material medium, their velocity may be different from c and the relation between wavelength λ, and frequency ν may be more complicated. In many materials the velocity of electromagnetic waves itself depends on the wavelength. An electromagnetic wave changes direction when going from one medium into another. The change of direction depends on the velocities in each medium. Since the wave velocity in a medium depends on the wavelength (colour), different colours break in different directions. This is how a glass prism produces the spectrum of different colours from "white" light.

23.5 Energy Radiated in Electromagnetic Waves

A charge moving at constant velocity forms constant (DC) currents and sets up constant magnetic fields. Time dependent, coupled magnetic and electric fields, electromagnetic waves, are NOT generated by charges moving at constant velocity.

Accelerated charges, however, do generate coupled, changing magnetic and electric fields. Electric and magnetic fields can affect other charges far away through Lorentz forces. Electromagnetic fields can accelerate charges, giving extra kinetic energy to charges. Thus electromagnetic waves carry energy through space. So an accelerated charge, in generating these electromagnetic fields, must be radiating, sending away, energy through space. The "power", the rate of energy radiated, is

$$P = \frac{dE}{dt} = \frac{1}{6\pi\epsilon_0}\frac{q^2}{c^3}a^2, \tag{23.29}$$

where q is the charge and a is its acceleration, c is the speed of light.

[4] 1 Angstrom $= 10^{-10}$ m; 1 nanometer $= 10^{-9}$ m

CHAPTER 23 - PROBLEMS:

1. If the two parallel plates we considered in this chapter extended from the $x = 0$ plane towards negative x, the waves would be traveling in the $-x$ direction. What is the form of the traveling wave solutions $E_z(x,t)$ and $B_y(x,t)$?

2. In a parallel plate arrangement, the separation between the two planes is 20 cm. An alternating voltage difference $V(t) = V_0 \cos(\omega t)$ of amplitude $V_0 = 3\ V$ is applied at $x = 0$.

   (a) What is the amplitude E_0 of the electric field oscillations? Edge effects are negligible, so that the electric field at $x = 0$ is in the z direction.

   (b) What is the amplitude B_0 of the magnetic field oscillations?

3. An alternating current $I(t) = I_0 \sin(\omega t)$ is running in an infinite straight wire. The alternating current frequency is $f = 50Hz$.

   (a) Sketch the time dependent magnetic and electric fields that this current sets up in space.

   (b) Sketch the directions of propagation of the electromagnetic waves.

   (c) What is the radian frequency ω of these electromagnetic waves?

   (d) What is the wavenumber k and the wavelength λ, assuming the waves are propagating in vacuum?

4. What are the frequencies and wavelengths of the radio waves carrying the signal from:

   (a) Your favorite FM radio stations?

   (b) A typical short wavelength radio station?

   (c) The electromagnetic waves used for TV broadcasts?

5. * **The Plasma Frequency** A plasma with n free electrons (and n ions of charge $+e$) per m^3 has a typical frequency of oscillation of the positive and negative charges against each other, called the *plasma frequency* ω_p:

$$\omega_p = \left(\frac{ne^2}{\epsilon_0 m_e}\right)^{1/2}$$

   where e is the charge and m_e the mass of the electron. Electromagnetic waves of frequency higher than the plasma frequency ω_p can pass through the plasma, while lower frequencies will be reflected.

   (a) The *interstellar medium* has an average electron density of 1 electron per cm^3. What is the plasma frequency of the interstellar medium?

   (b) What is the lowest frequency of electromagnetic waves that can propagate through the interstellar medium?

   (c) The ionosphere of the Earth is the plasma part of the atmosphere. Because the main source of ionization is due to Sun's ultraviolet radiation, the charge density of the ionosphere changes significantly for different altitudes and it also changes quite a lot during the day. Dayside ionospheric charge densities go up to $n = 10^6$ electrons per cm^3.

What is the lowest frequency that can penetrate the ionosphere with this density?

Radio waves of lower frequency are reflected by the ionosphere. Radio broadcasts use the ionosphere as a mirror to reach remote parts of the Earth by reflection.

For communication with space shuttles it is necessary to use signals above the plasma frequency of the ionosphere. When the space shuttles are passing through the ionosphere on their way back to Earth, the intense heat generation around the shuttle results with an increase in plasma density and therefore the plasma frequency. This causes a temporary communication black-out. The communication is recovered when the shuttle enters the neutral atmosphere[5].

[5]Francis Chen, *Introduction to Plasma Physics and Controlled Fusion*, 2nd Edition, Plenum Press, 1984.

Chapter 24

* Circuits

In this Chapter we will consider the potential energy per unit charge, the voltages, set up by source charge distributions, and the relations between voltage and charge and current distributions. Any charge distribution is a *circuit*, although typically one thinks of a device of some metal wires, in which electrons flow against resistance (ie friction) from other electrons and atoms in the wire. Such an electronic circuit is a good model for the kinds of things that can happen in a general circuit in a plasma, in a solution, or across the membrane of a cell in your body.

The metal wires in a circuit are material pieces where charges move against frictional resistance are the *resistors.*

There are gaps, empty space (or air or other materials), whose function is to keep charges separated and therefore to sustain electric fields related to the source charges by Gauss' Law. These charge distributions with gaps are the *capacitors.* If currents in the same direction keep bringing charges to opposite sides of a capacitor gap, eventually there will be a breakdown, a sudden discharge of the accumulated charge. Capacitors are therefore time dependent. In engineering designed circuits, alternating currents are used to avoid discharges. The charges and currents are alternating between limiting values that are sustainable without discharges. The electric fields in the capacitor gaps are then time dependent, so a capacitor also sustains magnetic fields induced by the displacement current term in the generalized Ampere's Law. The parallel plate capacitor is the simplest geometry for studying the properties of capacitance.

Moving charges (currents) induce magnetic fields (Ampere's Law). Currents are usually time dependent. In other words, moving charges are usually accelerated in most natural or engineering circuits. The induced magnetic fields are therefore also time dependent. By Faraday's Law, the time dependent magnetic fields induce electric fields. Pieces of circuit where there are time dependent currents sustain induced voltages (induced emf). These are the *inductors.* The simplest magnetic field geometry, that of a bar magnet, is provided by a solenoid. With time dependent currents, the solenoid is the simplest example of an inductor.

Every piece of a circuit actually contains resistive, capacitive and inductive properties simultaneously. The parallel plates in Chapter 23 sustained both electric and magnetic fields, so they have capacitance and inductance. Although we considered ideal conducting plates for simplicity, real materials have some resistance also. A circuit element designed to propagate electromagnetic waves, like the parallel plates in Chapter 23, is a *waveguide.* The *coaxial cable* that brings cable TV to your home is another example of a waveguide. The transmission properties of a waveguide, how transparent, absorbing or reflecting it is to electromagnetic waves of given frequencies, is called its *impedance*. Impedance depends on a combination of the capacitance, inductance and resistance properties of the material walls and the material in and around the waveguide, air, vacuum or whatever. A finite length waveguide, say a metal box with all sides closed, can also sustain standing electromagnetic wave patterns, depending on the way energy is given, voltages are applied, on its boundaries. Such a circuit element is called a *cavity* or *resonator*.

A circuit may have certain special *natural* frequencies at which the charge and current distributions oscillate in a special pattern. The natural frequencies depend on the circuit's capacitance, inductance and resistance properties. When oscillating at these natural frequencies, in its so called *normal* modes, the circuit radiates electromagnetic waves whose frequencies are the natural frequencies of the circuit. Conversely, if electromagnetic waves at those special natural frequencies are received by the circuit, it will efficiently absorb the energy in those waves, and the corresponding special pattern of oscillating charges and currents is excited. This is called resonance.

Specially designed circuit elements using fundamental quantum mechanical properties of matter are at the heart of modern technology. A very important example is the transistor whose resistance properties can be manipulated by voltage signals. Integrated circuits that contain large numbers of transistors, capacitors, inductors and resistors have been developed to a very high level of complexity in integration and a very small scale of the individual parts. The understanding of quantum mechanical material and electromagnetic properties of inorganic and biological "circuit elements" at the nano scale ($10^{-9}\ m$) is a very active area of research in both fundamental science and in applied science and engineering.

The time dependent charge and current distributions act as sources for time dependent coupled electric and magnetic fields which in turn effect the local charges and currents through the Lorentz force. The time dependent fields propagate away as electromagnetic waves, as we saw in Chapter 23. These electromagnetic fields can do work on charges far way from the original circuit that generated the electromagnetic waves. *Electromagnetic fields can store and carry energy.* The ways of storing and spending energy in circuits, and in their electromagnetic fields, can be understood through a discussion of the simplest resistive, capacitive and inductive circuit elements and the voltages they sustain.

24.1 Ohm's Law

Consider a particle of charge q and mass m moving in an electric field $\mathbf{E}$. If this charge is moving in vacuum, there is nothing but the electric field to influence it. From Newton's 2^{nd} Law the instantaneous acceleration is

$$\mathbf{a}(t) = \frac{q\mathbf{E}(\mathbf{r}(t))}{m}$$

where $\mathbf{E}(\mathbf{r}(t))$ is the electric field at the instantaneous position $\mathbf{r}(t)$ of the charge.

In the real world a charge is always in a material medium, with other charged or neutral particles around, with which it interacts. Electrons flowing in copper wire or ions in a solution or plasma collide with other particles all the time. Charges accelerate in the electric field, but they also lose the velocity acquired in collisions. The average effect of all the collisions is represented by a friction force which opposes the motion, converting the kinetic energy of motion along the direction of the field, acquired by the work done by the electric field, into kinetic energy of motion in random directions, namely heat.

A common, but by no means universal, approximate form of the friction force, which occurs in many systems, including common circuits, is:

$$\mathbf{F}_f = -\alpha\mathbf{v}$$

Without an electric field, such a friction force, linear in the velocity, leads to simple exponential damping, as we saw in Chapter 14:

$$\mathbf{F}_f = -\alpha\mathbf{v} = m\frac{d\mathbf{v}}{dt}$$
$$\frac{d\mathbf{v}}{dt} = -\frac{\alpha}{m}\mathbf{v} \equiv -\frac{\mathbf{v}}{\tau}$$

where $\tau \equiv m/\alpha$ is called the damping time. The physical meaning of τ is the average time between collisions.

Including both the electric force and the friction force in the equation of motion, one obtains

$$\mathbf{F}_{total} = q\mathbf{E} - \alpha\mathbf{v} = m\mathbf{a} = m\frac{d\mathbf{v}}{dt}. \tag{24.1}$$

The forces will balance, to give a "*steady state*" constant velocity

$$\mathbf{v}_0 = \frac{q\mathbf{E}}{\alpha} = \frac{q\mathbf{E}\tau}{m}. \tag{24.2}$$

In a wire with electrons carrying current the average velocity $\langle\mathbf{v}\rangle$ of the electrons is

$$\langle\mathbf{v}\rangle = \frac{-e\mathbf{E}\tau}{m}. \tag{24.3}$$

If there are n moving electrons per unit volume, the current density $\mathbf{j}$ is

$$\mathbf{j} = n(-e)\langle\mathbf{v}\rangle = \frac{ne^2\tau}{m}\mathbf{E} \equiv \sigma\mathbf{E}. \tag{24.4}$$

this is the basic statement of *Ohm's Law.* The current density is proportional to the electric field. The proportionality constant $\sigma \equiv ne^2\tau/m$ is the *conductivity* of the material medium.

The total current I through a cross sectional area A, say for example of a copper wire, is:

$$I = jA, \tag{24.5}$$

and the voltage difference V over a length l of wire in the direction of the electric field is

$$V = El. \tag{24.6}$$

From the last three equations one obtains the macroscopic, or circuit, form of Ohm's Law,

$$V = \frac{l}{\sigma A}I \equiv \frac{\rho\, l}{A}I \equiv RI \tag{24.7}$$

where $\rho \equiv \sigma^{-1}$ is the *resistivity* and R is the *resistance.* The SI unit of resistance is the Ohm, in honour of the German physicist Ohm:

$$1 \text{ Ohm } (\Omega) = 1 \text{ Volt/Ampere.}$$

Note again that Ohm's Law is valid only in situations where the friction force is *linear* in the velocity. If the friction force has a nonlinear dependence on velocity, which does happen in many circuits, there is a steady state value of the average velocity and current, but this gives a nonlinear relation between $\mathbf{j}$ and $\mathbf{E}$ and between V and I, relations ("*current-voltage characteristics*") that are different from Ohm's Law. The simple linear case of Ohm's Law does apply in many practical circuits.

24.2 Work, Energy, Power

The bit of work dW_E done by the electric field in moving a tiny charge dq from a point P_1 to a point P_2 is

$$dW_E = \int_1^2 \mathbf{F}_E \cdot \mathbf{dl} = dq \int_1^2 \mathbf{E} \cdot \mathbf{dl} = U(1) - U(2) = dq[V(1) - V(2)] = V dq \tag{24.8}$$

where V is the voltage difference of the initial point P_1 with respect to the final point P_2. On an amount of charge dq entering the circuit in time dt the electric field is doing work (*expending power*) at the rate:

$$\frac{dW_E}{dt} = V \frac{dq}{dt} = V I \tag{24.9}$$

between the points P_1 and P_2. The SI unit of power, rate of doing work, or rate of increase or decrease of energy, is the Watt[1].

$$1 \text{ Watt (W)} \equiv 1 \text{ Joule/sec (J/s).}$$

In the case of resistive elements, the work done by the electric field is positive while the work done by friction is negative, and of the same magnitude as the work done by the electric field. The average speed of the current carriers and the current they sustain remains constant. The collisions that we call friction collectively channel the work done by the electric field into microscopic kinetic energy of motion of the charges in random directions - in other words, *heat* is generated, or energy is *dissipated.* By the conservation of energy, heat is generated at the same rate as the rate the electric field, the power $P = V\ I$. If Ohm's Law is valid, the power for heat generation is

$$P = V I = \frac{V^2}{R} = I^2\ R. \tag{24.10}$$

24.3 Energy Stored in the Electric Field

Consider a parallel plate capacitor, having charges $\pm Q$ and charge densities $\pm\sigma$ on its plates of area A and separation D. With its uniform electric field of magnitude

$$E = \frac{\sigma}{\epsilon_0} = \frac{Q}{\epsilon_0\ A} \tag{24.11}$$

[1]in honour of Sir James Watt, who invented the advanced form of the steam engine that lead to the industrial revolution

and the voltage difference

$$V = E\ D = \frac{Q\ D}{\epsilon_0\ A} \tag{24.12}$$

between the plates, this is one of the simplest cases of an electric field set up by charges held in separation. The linear dependence of electric fields and voltages on the source charge leads to the definition of *capacitance* C, the proportionality constant between source charge and voltage, through the general relation

$$V \equiv \frac{Q}{C} \tag{24.13}$$

as discussed in Chapter 18. The parallel plate capacitor has the capacitance

$$C_{parallelplate} = \frac{\epsilon_0\ A}{D}, \tag{24.14}$$

as seen from Eq. 24.12.

The oppositely charged plates of the parallel plate capacitor attract each other with the Coulomb force. Forces other than this direct Coulomb force, typically material forces giving rigidity to materials supporting the plates, are employed to keep the plates apart. If there were no forces other than the Coulomb (electrostatic) force, the plates would accelerate towards each other, meet and neutralize. This means the capacitor has a positive amount of electrostatic potential energy, which would be converted to kinetic energy if the plates were released and accelerated towards each other under the influence of the Coulomb force.

The electrostatic potential energy of the two parallel capacitor plates at separation D can be calculated in a number of ways, by taking the charge configuration from an initial, reference configuration to the final separation D and calculating the work done by the electrostatic force in the process. The potential energy difference between the initial and final configurations does not depend on which path is followed in taking the system from the initial to the final configuration.

Let us take the initial configuration to be the one with positively charged plate lying on top of the negatively charged plate, at zero or with a very small separation from the negative plate. Now keep the negatively charged plate fixed and move the positively charged plate out to separation D, keeping it parallel to the negative plate throughout the motion, as shown in Fig. 24.1. The same amount of work is done *by the electric field due to the negatively charged plate* on every unit of charge on the positive plate. As we derived in Chapter 18, the uniform electric field due to the negatively charged plate is perpendicular to the plates, points towards the negatively charged plate and has the magnitude

$$E' = \frac{\sigma}{2\ \epsilon_0} = \frac{Q}{2\ \epsilon_0\ A}. \tag{24.15}$$

The total work done by the electrostatic force of the negatively charged plate, held fixed at $z = 0$ on the positively charged plate as the latter moves from $z = 0$ to $z = D$ gives the potential energy difference between the two configurations:

$$U(z = 0) - U(z = D) = \int_0^D \mathbf{F}' \cdot \mathbf{dz}. \tag{24.16}$$

The infinitesimal "step" displacement vector is in the z direction, $\mathbf{dz} = dz\ \hat{\mathbf{k}}$ perpendicular

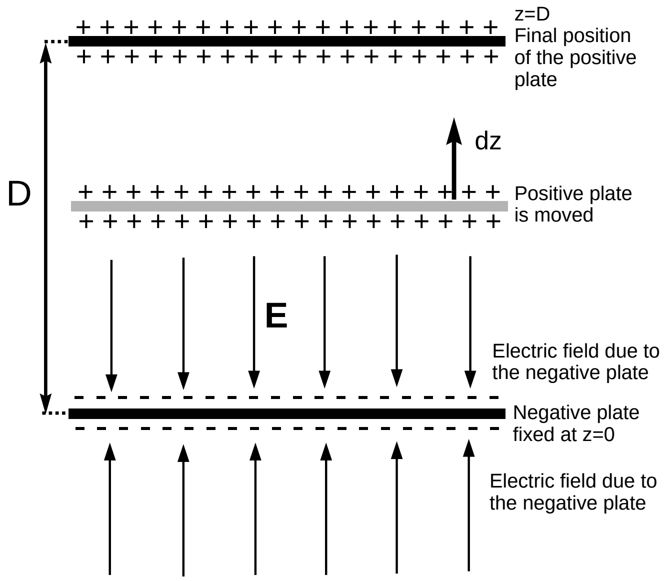


Figure 24.1:

to the plates. The electrostatic force $\mathbf{F}'$ exerted by the negatively charged plate on the positively charged plate with charge $+Q$ is

$$\mathbf{F}' = +Q\mathbf{E}' = +Q(-E'\hat{\mathbf{k}}).$$

Taking the reference potential energy to be zero, $U(z=0)=0$, we find

$$U(z=D) = QE'D = \frac{Q^2\ D}{2\ \epsilon_0\ A} = \frac{Q^2}{2\ C} = \frac{1}{2}\ CV^2 \tag{24.17}$$

using the definition of capacitance, Eq. 24.13 relating the charge Q and the voltage V, and the value of the capacitance for the parallel plate system, given in Eq. 24.14

This potential energy is there because of the charge separation, because of the electric field set up by the charges. The energy can be expressed in terms of the total electric field magnitude

$$E = \frac{V}{D} = \frac{Q}{\epsilon_0\ A}$$

between the plates, as

$$U = \frac{1}{2}\ CV^2 = \frac{1}{2}\ \epsilon_0\ E^2\ A\ D. \tag{24.18}$$

This means that the energy stored in the electric field has an *energy density* u_E, an energy per unit volume,

$$u_E = \frac{1}{2}\ \epsilon_0\ E^2. \tag{24.19}$$

It can be shown that this expression for energy density in electric fields in vacuum is valid generally, it is not restricted to the parallel plate capacitor.

24.4 Energy Stored in the Magnetic Field

Consider a solenoid with current I running through its coils wound with N windings along the length h of the solenoid ($n = N/h$ windings per unit length). The magnetic field inside the solenoid is parallel to the solenoid's axis, and has the value

$$B = \mu_0 n I \tag{24.20}$$

as derived in Problem 6 of Chapter 19.

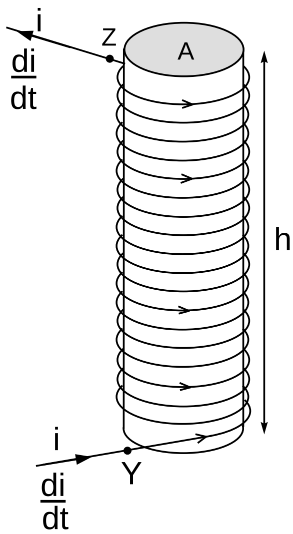


Figure 24.2:

How much energy is involved in building up this configuration of the solenoid, with current I running through its coils? Take the initial configuration to be with no current. Imagine the current is increased at a rate di/dt. According to Faraday's Law, the changing current, and the resulting change in the magnetic field through the solenoid will induce a voltage through the wire. The direction of this voltage is opposed to the direction of di/dt, and hence to the direction of the final current I. Taking the direction of I and di/dt, from point Y to point Z in Figure 24.2, as positive, the voltage difference between these two points is

$$V \equiv V(Z) - V(A) = -L\,\frac{dI}{dt}. \tag{24.21}$$

The *inductance* L for the solenoid is

$$L_{solenoid} = \mu_0 n N A = \mu_0\; n^2\; A\; h \tag{24.22}$$

where A is the cross sectional area of the solenoid (see Problem 6 of Chapter 19).

During the process of building up the current in the solenoid many charges dq have moved through the solenoid, from point Y to point Z. The induced electric field in the coil is in the direction from Z to Y. If there were no other forces, the induced electric field would accelerate the charges dq back towards Y. The work done by the induced electric field is

$$dW_{E,ind} = V dq = -L\frac{di}{dt}\, dq = -L\ i\ di$$

noting that dq/dt is just the instantaneous current i.

The motion of the charges dq up along the solenoid from Y to Z is due to forces other than the induced electric field, for example electric fields caused by far sources, like a battery in another part of the circuit. The work done by these other forces is just the opposite of the work done by the induced electric field, so that the increase in the energy of the solenoid is

$$dW = L\ i\ di$$

leading to a total energy stored in the solenoid as the current is built up from 0 to I

$$\mathcal{E} = \int_0^I dW = \int_0^I L\ i\ di = \frac{1}{2} L\ I^2. \tag{24.23}$$

This stored energy can be expressed in terms of the magnetic field strength B inside the solenoid using Eq. 24.20 and Eq. 24.22 as

$$\mathcal{E} = \frac{1}{2} L\ I^2 = \frac{1}{2}\mu_0\ n^2\ A\ h\ I^2 = \frac{1}{2\ \mu_0}\ B^2\ A\ h. \tag{24.24}$$

The energy stored per unit volume is

$$u_B = \frac{1}{2\ \mu_0}\ B^2 \tag{24.25}$$

where u_B is the energy density in the magnetic field. This energy density is stored in any magnetic field, and the expression is not limited to the solenoid.

24.5 Energy in Electromagnetic Waves

In the space between the infinite parallel plates discussed in Chapter 23, electromagnetic waves are running with the electric and magnetic components

$$\begin{aligned} E_z(x,t) &= E_0 \cos(kx - \omega t) \\ B_y(x,t) &= B_0 \cos(kx - \omega t) \\ B_0 &= -\frac{E_0}{c}. \end{aligned}$$

The energy density in the electric field at time t, and at points at distance x from the edge is

$$u_E(x,t) = \frac{1}{2}\ \epsilon_0\ E(x,t)^2 = \frac{1}{2}\ \epsilon_0\ E_0^2 \cos^2(kx - \omega t). \tag{24.26}$$

The time averaged value of this is

$$< u_E >= \frac{1}{2}\ \epsilon_0\ E_0^2 < \cos^2(kx - \omega t) >= \frac{1}{4}\ \epsilon_0\ E_0^2, \tag{24.27}$$

since the time averaged value $< \cos^2(kx - \omega t) >= 1/2$. Thus the time averaged energy density in the electric field has the same value, independently of x at all points between the semi-infinite parallel plates of the waveguide.

The time averaged magnetic energy density in the electromagnetic waves can be calculated in similar fashion. The energy density in the magnetic field at time t, and at points at distance x from the edge is

$$u_B(x,t) = \frac{1}{2\,\mu_0}\; B(x,t)^2 = \frac{1}{2\,\mu_0}\; B_0^2 \cos^2(kx - \omega t). \tag{24.28}$$

The time averaged value of this is

$$< u_B >= \frac{1}{2\,\mu_0}\; B_0^2 < \cos^2(kx - \omega t) >= \frac{1}{4\,\mu_0}\; B_0^2. \tag{24.29}$$

Thus the time averaged energy density in the magnetic field has the same value, independently of x at all points between the semi-infinite parallel plates of the waveguide. Moreover, since $B_0 = -E_0/c$ and $1/c^2 = \epsilon_0\,\mu_0$, the average energy densities in the electric and magnetic fields are equal,

$$< u_B >=< u_E >= \frac{1}{4}\;\epsilon_0\; E_0^2. \tag{24.30}$$

The result that the average electric and magnetic energy densities are equal is true for all electromagnetic waves, no matter what their pattern of propagation is. The total time averaged energy density, in both the electric and the magnetic fields is

$$< u_{EM} >=< u_B > + < u_E >= \frac{1}{2}\;\epsilon_0\; E_0^2 = \frac{1}{2\,\mu_0}\; B_0^2. \tag{24.31}$$

As we saw in Chapter 23 with the example of electromagnetic wave propagation between the semi-infinite parallel plates, the electric and magnetic field directions are perpendicular to each other and to the direction of propagation of the waves, such that the vector product $\mathbf{E} \times \mathbf{B}$ points in the direction of propagation. Multiplying with appropriate constants, the *Poynting vector* $\mathbf{S}$ is defined such that it has dimensions of energy flux, energy per unit area per unit time, and is directed in the same direction as the direction of propagation of the electromagnetic waves:

$$\mathbf{S}(x,t) \equiv c\;\left(\epsilon_0^{1/2}\;\mathbf{E}(x,t) \times \frac{\mathbf{B}(x,t)}{\mu_0^{1/2}}\right). \tag{24.32}$$

The time averaged value of the Poynting flux is found to be

$$< \mathbf{S} >= c\;\left(\frac{\epsilon_0\;E_0^2}{2}\right)\hat{\mathbf{i}} =< u_{EM} >\;\; c\,\hat{\mathbf{i}} \tag{24.33}$$

which is precisely the energy per unit area per unit time transported across any area perpendicular to the direction of propagation $\hat{\mathbf{i}}$ of the electromagnetic waves. The Poynting vector as defined in Eq. 24.32 represents the energy flux transported by the electromagnetic waves quite generally, no matter what the pattern of electromagnetic radiation is.

CHAPTER 24 - PROBLEMS:

1. **Currents and charges in the semi-infinite parallel plate waveguide of Chapter 23**
   (a) Use Gauss' Law to infer the charge density $\sigma(x,t)$ on the plates from the electric field $E_z(x,t)$ of the electromagnetic wave.
   (b) Use Ampere's Law to infer the current density $\mathbf{J}(x,t)$ (its direction and units) on the plates from the magnetic field $B_y(x,t)$ of the electromagnetic wave.
   (c) Use the relation between the amplitudes of the electric and magnetic fields, E_0 and B_0, to express both $\sigma(x,t)$ and $\mathbf{J}(x,t)$ in terms of E_0.
   (d) Sketch $\mathbf{J}(x,t)$ and $\sigma(x,t)$.
   (e) The local conservation of charge states that the charge on any area element of the plates can only increase (decrease) if there is more (less) current moving into the area than leaving it. Express this mathematically as a relation between $\mathbf{J}(x,t)$ and $\sigma(x,t)$, and check if your results for $\mathbf{J}(x,t)$ and $\sigma(x,t)$ satisfy local charge conservation.

2. **Resistance in the semi-infinite parallel plate waveguide of Chapter 23** The parallel plates in the semi-infinite waveguide are real conductors, so they have some small resistivity. What is the effect of resistivity? Describe *qualitatively* what happens to the currents, charge densities and the amplitudes of the electromagnetic fields in the wave. Energy is fed into the system all the time through the external voltage $V(t) = V_0 \cos(\omega\ t)$. Does this energy arrive at the other end of the waveguide, at infinity? What happens to the energy?

3. **Basic Circuits** The voltage difference between two points A and B, or the work done by the electric fields when a unit charge is moved from A to B, does not depend on which path is followed from A to B. In any closed circuit, one can choose two points A and B, separating the circuit into two different paths, say the *left* and *right* paths, both leading from A to B. Equating the voltage difference from A to B through the different circuit elements on the two paths gives an equation

$$V_{AB,left} = V_{AB,right}$$

   Equivalently the voltage difference *around the circuit*, from A to B, say on the left path, and then from B back to A on the right path, is zero:

$$V_{AB,left} + V_{BA,right} = V_{AB,left} - V_{AB,right} = 0$$

   Such circuit equations contain resistance, capacitance and inductance parameters of the circuit and relate charges, currents and their time derivatives, thus describing the dynamics of the circuit.

   (a) The circuit has an inductance L, with $V_{AB,left} = -LdI/dt$, and a capacitance C, with $V_{AB,right} = Q/C$ (Figure 24.3 (a)). Write the circuit equation in terms of $Q(t)$ and its derivatives, using $I = dQ/dt$. What is the solution for $Q(t)$ and

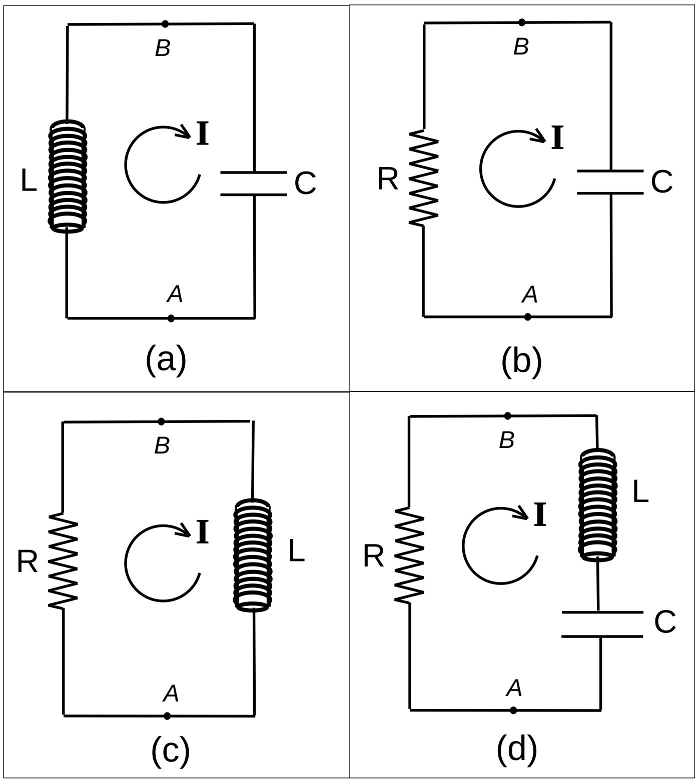


Figure 24.3: The arrows indicate our *choice* of (+) direction of current. Negative and AC currents also run in these circuits.

$I(t)$ if initially charges $\pm Q_0$ are placed, at rest, on the capacitor plates, with no current: the initial conditions are $Q(t=0) = Q_0$ and $I(t=0) = 0$.

(b) The circuit has a resistance R, with $V_{AB,left} = -RI$, and a capacitance C, with $V_{AB,right} = Q/C$ (Figure 24.3 (b)). Write the circuit equation in terms of $Q(t)$ and its derivative, using $I = dQ/dt$. What is the solution for $Q(t)$ and $I(t)$ if initially charges $\pm Q_0$ are placed on the capacitor plates: the initial condition is $Q(t=0) = Q_0$.

(c) The circuit has a resistance R, with $V_{AB,left} = -RI$, and an inductance L, with $V_{AB,right} = LdI/dt$ (Figure 24.3 (c)). Write the circuit equation in terms of $I(t)$ and its derivative. What is the solution for $I(t)$ if the initial current has the value $I(t=0) = I_0$?

(d) ** *This item requires using (learning) complex numbers.* The circuit has a resistance R, with $V_{AB,left} = -RI$, a capacitance C, and an inductance L, such that $V_{AB,right} = Q/C + L\,dI/dt$ (Figure 24.3 (d)). Write the circuit equation in terms of $Q(t)$ and its derivatives, using $I = dQ/dt$. What is the solution for $Q(t)$ and $I(t)$ if the initial conditions are $Q(t=0) = Q_0$ and $I(t=0) = I_0$? Hint: Try a solution of the form $Q(t) = Q_0\ \exp(\lambda\, t)$. For some circuits you will find that the parameter λ is a *complex number*. What are the conditions for λ to be *real*? *complex*? *pure imaginary*? What are the solutions for $Q(t)$ and $I(t)$ like in each of these cases?

Exam Problems - Electricity and Magnetism

1. **[Spring 2007, Final]** A particle of mass m and charge $+q$ is moving in a uniform magnetic field in the z direction, $\mathbf{B} = B\mathbf{k}$. In addition, there is a uniform electric field, also in the z direction, $\mathbf{E} = E\mathbf{k}$. The velocity of the particle at time $t = 0$ is given by $v(0) = v\mathbf{i} + u\mathbf{k}$.
The unit vectors in the x, y, z directions are denoted by $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, respectively. The Lorentz force is given by $\mathbf{F} = q(\mathbf{E} + \mathbf{v}(t)\mathbf{B})$.

   (a) What is the acceleration vector $\mathbf{a}(0)$ at $t = 0$?

   (b) What kind of trajectory does the particle follow $-$ make a drawing.

   (c) What is the angular frequency ω of the circular motion in the x-y plane?

   (d) What is the radius of the motion in the x-y plane?

2. **[Fall 2006, Final]**

   (a) Figure 24.4(a) shows a solenoid of length Z. The current I is constant. Sketch the magnetic field lines.

   (b) Figure 24.4(b) shows an infinite solenoid. In the approximation of an infinite solenoid, *the magnetic field outside the solenoid is zero.* Use Ampere's Law, with the closed loop shown, to find an expression for the strength of the magnetic field inside the solenoid.

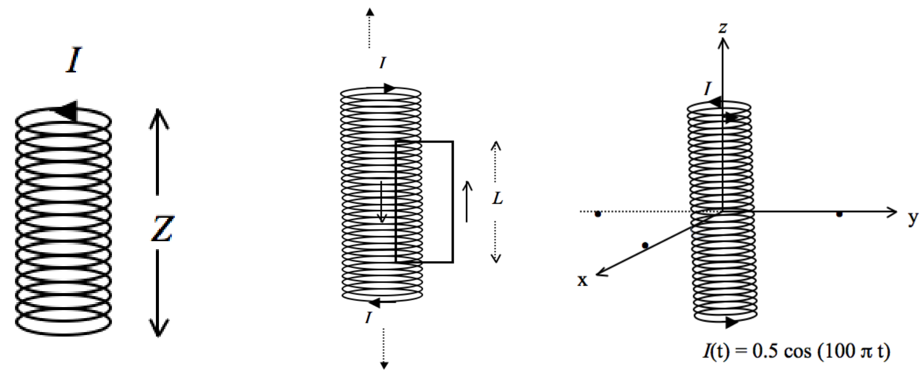
   (c) Use your result in (b) to calculate the value of the magnetic field, in SI units (Tesla), inside an infinite solenoid with 10^7 windings, per meter, of the current carrying wire, and a constant current $I = 1/4\pi\ Ampere$. Given: $\mu_0 = 4\pi \times 10^{-7}$ $(Tesla \cdot meter/Ampere)$.

   (d) A time dependent current $I(t) = 0.5\cos(100\pi t)$ is running in the infinite solenoid. Sketch, on Figure 24.4(c), the directions of electric and magnetic fields and the directions of travel of electromagnetic waves at the points indicated.

3. **[Fall 2009, Final]** A uniform magnetic field $\mathbf{B}$ lies in the x direction as shown in the Figure 24.5. A small copper circle with radius a rotates about the y-axis with angular frequency ω.

   (a) Find the magnetic flux threading the circle as a function of time.

   (b) Find the "electromotive force" (voltage) induced around the circle as a function of time.



(a) (b) Infinite Solenoid (c) Infinite Solenoid

Figure 24.4:

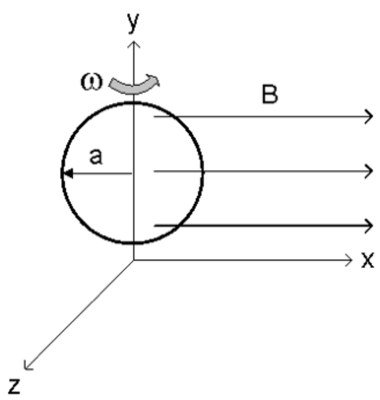


Figure 24.5:

4. **[Fall 2008, Final]** Two parallel half-infinite conducting planes have a time dependent voltage difference $V(t) = A\cos\omega t$ at the edge (see Figure 24.6).

   (a) Sketch the electric field directions in between the planes.

   (b) Sketch the magnetic field directions in between the planes.

   (c) Sketch the direction of propagation of electromagnetic waves in between the planes.

   (d) What is the wavelength of the electromagnetic wave if the frequency is $f = \omega/2\pi = 3\times 10^6\,s^{-1}$?
   Given: the speed of light $c = 3\times 10^8\,m/s$.

5. **[Fall 2007, Final]** Figure 24.7 shows a coaxial cable, made of a uniform metal wire of radius a, and an infinitely long thin conducting cylindrical shell with radius b. A constant electric current I is running though the wire, with uniform current density $j = I/\pi a^2$. A constant electric current $-I$ flows along the surface of the coaxial shell. Using Ampere's Law, find the magnitude and show the direction of the magnetic field at a point at distance r from the axis,

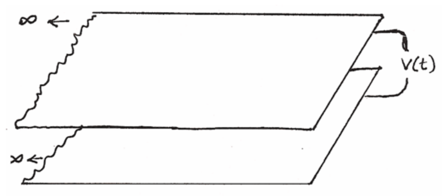


Figure 24.6:

(a) for $r < a$

(b) for $a < r < b$

(c) for $r > b$.

(d) If the currents I and $-I$ are not constant but time dependent (alternating currents), the coaxial cable will radiate electromagnetic waves. Sketch the directions of travel of these waves on the figure.

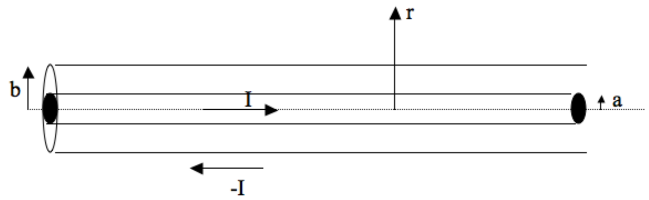


Figure 24.7:

6. **[Fall 2003, Final]** Consider an infinitely long, straight wire.

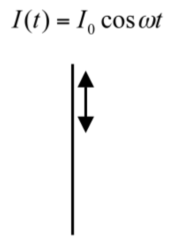


Figure 24.8:

(a) If the wire has constant positive charge density λ ($Coulombs/m$), determine the electric field (magnitude and direction) at distance r from the wire using Gauss' law.

(b) If the wire is neutral but carrying a constant current I, determine the magnetic field (magnitude and direction) at distance r from the wire using Ampere's law.

(c) Now consider what happens if the current is alternating, $I(t) = I_0 \cos \omega t$ (see Figure 24.8). Sketch the directions of the electric and magnetic fields and the directions of propagation of the electromagnetic fields in the space around the wire.

7. **[Spring 2007, Final]** An alternating current $I(t) = I_0 \sin(\omega t)$ is running in an infinite straight wire (see Figure 24.9). The alternating current frequency is $f = 50\,Hz$.

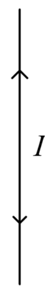


Figure 24.9:

(a) Sketch on the figure the time dependent magnetic and electric fields that this current sets up in space.

(b) Sketch the directions of propagation of the electromagnetic waves.

(c) What is the radian frequency ω of these electromagnetic waves?

(d) What is the wavenumber k and the wavelength λ, assuming the waves are propagating in vacuum? The speed of light in vacuum is $c = 3 \times 10^8\,m/s$.

8. **[Spring 2010, Final]** An electron travels with a constant speed v_e between two charged parallel plates that are separated by D and charged by voltage V as shown in Figure 24.10.

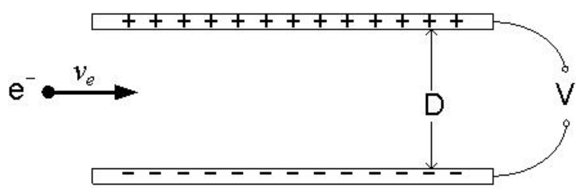


Figure 24.10:

(a) What is the electric field **E** between the two plates?

(b) What is the electric force **F** on the electron?

(c) What magnetic field **B** should be applied so that the electron would not be deflected?

[Remember: A vector has magnitude and direction]

9. **[Fall 2010, Midterm 2]** Consider an infinitely-long coaxial cable, consisting of a solid cylindrical conductor with radius a and a thin outer conductor with inner radius b and outer radius c, as shown in Figure 24.11. The current $+I$ flows through the inner conductor, uniformly spread across the conductor. The current $-I$ flows uniformly across the outer conductor.

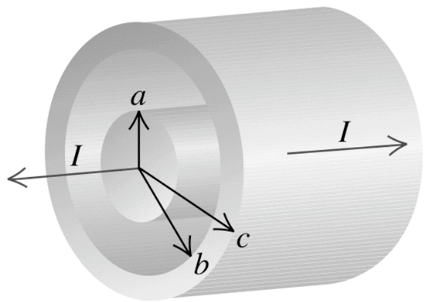


Figure 24.11:

Find the magnetic field vector $\mathbf{B}(r)$ (magnitude and direction) for the following cases:

(a) $r < a$
(b) $a < r < b$
(c) $b < r < c$
(d) $r > c$
(e) Sketch $B(r)$ as a function of r, from $r = 0$ to $r > c$.

10. **[Fall 2010, Midterm 2]** Three point charges reside on the corners of equilateral triangle with side length of a, as shown in Figure 24.12.

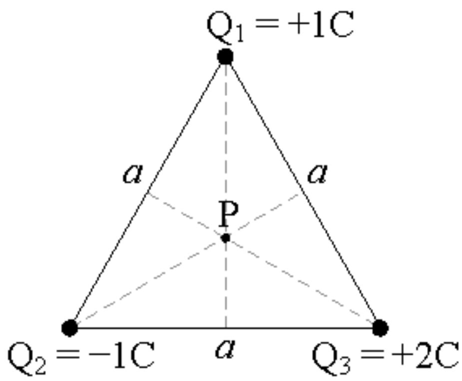


Figure 24.12:

(a) Find the electric force $\mathbf{F}_1$ on charge Q_1.
(b) Find the electric force $\mathbf{F}_2$ on charge Q_2.

(c) Find the electric force $\mathbf{F}_3$ on charge Q_3.

(d) Find the electric field $\mathbf{E}_{\mathrm{P}}$ at the center of the triangle (at point P).

11. **[Fall 2010, Midterm 2]** The magnetic field in the space is given as $B = 0.001$ Tesla oriented along the $+z$ direction. A proton is suddenly generated with an initial velocity of $v_0 = 1,000,000\,\mathrm{m/s}$ along the $+x$ direction (see Figure 24.13). [$e = 1.6 \times 10^{-19}\,\mathrm{C}$, $m_p = 1.6 \times 10^{-27}\,\mathrm{kg}$]

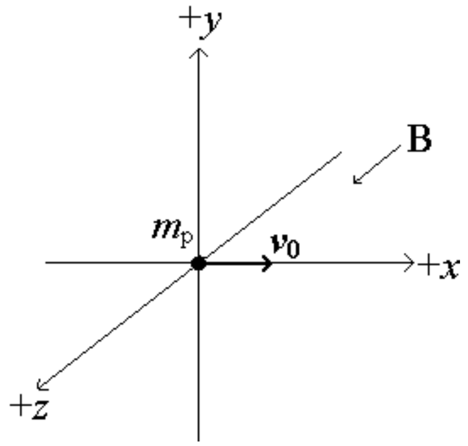


Figure 24.13:

(a) Find the force on the proton due to **B** when it first appears with v_0.

(b) What would be the motion and the path of the proton? Describe the motion, find relevant parameters, and draw the path of the proton on the figure.

12. **[Fall 2010, Midterm 2]** Consider two infinitely large charged sheets with charge densities $+\sigma$ and $-\sigma$. They are oriented perpendicular to each other, as shown in Figure 24.14. Draw the electric field pattern everywhere around the charged sheets.

[Spring 2011, Midterm 2] The parallel plate capacitor shown in Figure 24.15 has area $A = 10^{-2}\,\mathrm{m}^2$ and separation $d = 10^{-3}\,\mathrm{m}$. The plates carry charges $Q = +9 \times 10^{-4}$ Coulomb and $-Q = -9 \times 10^{-4}$ Coulomb. Take $\epsilon_0 = 9 \times 10^{-12}\,Coulomb^2\,m^{-2}\,N^{-1}$.

(a) What is the charge density σ in $Coulomb\,/\,m^2$?

(b) We found, using Gauss' Law, that the magnitude of the electric field between the capacitor plates, far from the edges, is given by $E = \sigma/\epsilon_0$. Find E, give its units and show the direction of **E** on the figure.

(c) What is the voltage difference V between the capacitor plates?

(d) In charged systems the voltage V is always proportional to the source charge Q, such that $Q = CV$. The proportionality constant C is called the capacitance. Its SI unit is $Coulomb\,/\,Volt$, alternatively called the $Farad$. Find the capacitance of this system in $Farads$.

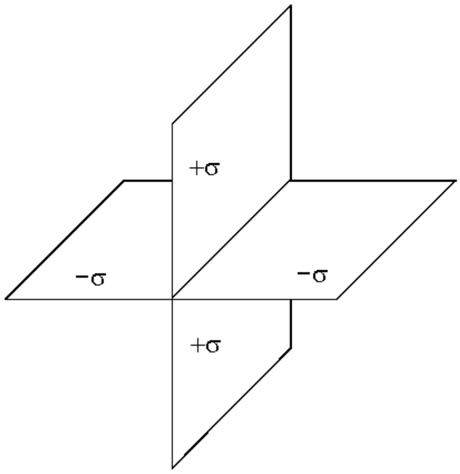


Figure 24.14:

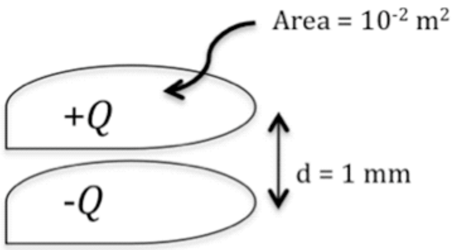


Figure 24.15:

[Spring 2011, Midterm 2] Two very long parallel wires separated by 2 m each carry a current of 0.5 Ampere in the same direction, coming out of the page (shown with ⊙) in Figure 24.16A. $\mu_0 = 4 \times 10^{-7}\, kg\, m\, C^{-2}$.

(a) Find the magnitude of the magnetic field $\mathbf{B}_1$ due to wire 1 at the midpoint P_1. Show the direction of $\mathbf{B}_1$ on the figure.

(b) Find the magnitude of the magnetic field $\mathbf{B}_2$ due to wire 2 at the midpoint P_1. Show the direction of $\mathbf{B}_2$ on the figure.

(c) What is the magnitude of the total magnetic field $\mathbf{B}$ at P_1?

(d) Show the direction of the total magnetic field $\mathbf{B}$ at P_2 on the figure.

(e) Now suppose the current in wire 2 is stopped. A ball with charge 1 C is shown with q, moving with speed $v = 10\,\mathrm{ms}^{-1}$ in the direction shown in Figure 24.16B. What is the magnitude and direction of the force on the ball?

[Spring 2011, Midterm 2] A toroid is a donut or simit shaped solenoid as shown in the top panel of Figure 24.17, with the wire carrying current I winding about the toroid N times. The bottom panel in Figure 24.17 show the toroid in cross section, ⊙ indicating current out of page and ⊗ indicating current into the page.

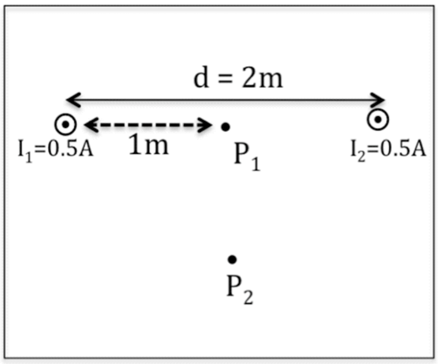


Figure A

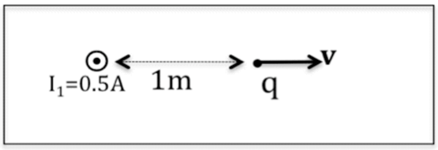


Figure B

Figure 24.16:

(a) By symmetry, the magnetic field strength should be the same everywhere on the circle C. Show the direction of the magnetic field at the points P_1 and P_2 on the figure.

(b) Use Ampere's Law to find the magnetic field strength on the circle C in terms of N, I, and the radius R of this circle.

(c) Find the value of the magnetic field if the number of turns $N = 10^4$, $I = 1\,A$ and $R = 2\,\text{m}$. $\mu_0 = 4 \times 10^{-7}\, kg\, m\, C^{-2}$.

(d) What is the magnetic field strength at the point P_3?

(e) What is the magnetic field strength at the point P_4?

13. **[Spring 2011, Midterm 2]** A particle of mass m and charge $+q$, moving with velocity v is initially at point P at distance R from an infinite straight wire carrying charge per unit length $+\lambda\, Coulomb\,/\,m$. There is also a uniform magnetic field **B** parallel to the wire, as shown in Figure 24.18. The particle starts to move on the circle with radius R centered on the wire, with uniform speed v.

   (a) Draw the electric field lines on the figure.

   (b) What is the magnitude of the electric field at any point P' on the circle of radius R? What is the magnitude and direction of the electric force on the charge $+q$?

   (c) What is the magnitude and direction of the magnetic force at any point P' on this circle?

   (d) Write Newton's 2nd Law for this motion, equating the total electric and magnetic force to ma for acceleration in uniform circular motion.

   (e) What is the charge density λ on the wire needed to have uniform circular motion with the given speed v on this circle of radius R?

14. **[Spring 2011, Final]** An alternating current $I(t) = I_0 \sin \omega t$ is running in an infinite straight wire (see Figure 24.19). The alternating current frequency is $f = 50\,\text{Hz}$.

   (a) Sketch on the figure the time dependent magnetic fields that this current sets up in space.

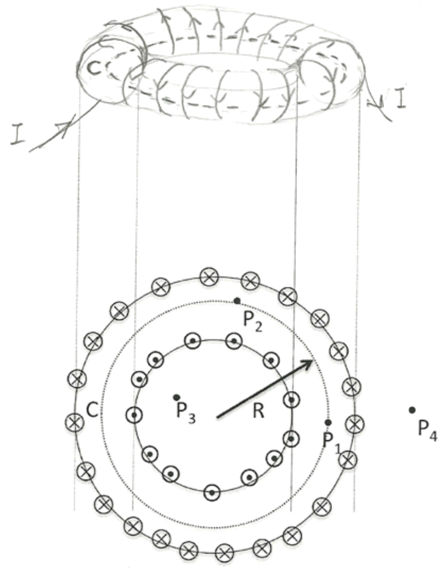


Figure 24.17:

(b) Sketch the time dependent electric fields.

(c) Sketch the directions of propagation of the electromagnetic waves.

(d) What is the wavelength λ of these electromagnetic waves? Given: $c = 3 \times 10^8$ m/s

15. **[Spring 2011, Final]** A particle of charge q and mass m is thrown into a uniform magnetic field $\mathbf{B}$ with an initial velocity vector $\mathbf{v}_0$ lying in a horizontal plane perpendicular to the magnetic field, as shown in Figure 24.20. There is no friction.

(a) What is the direction and magnitude of the acceleration $\mathbf{a}$?

(b) What is the angle between the acceleration and the velocity?

(c) What is the shape of the orbit that the particle will follow?

(d) Write Newton's Second Law for this motion and solve it to find the angular velocity ω of the particle.

(e) Does the kinetic energy change during this motion? How much work is done on the particle during one period of its orbit?

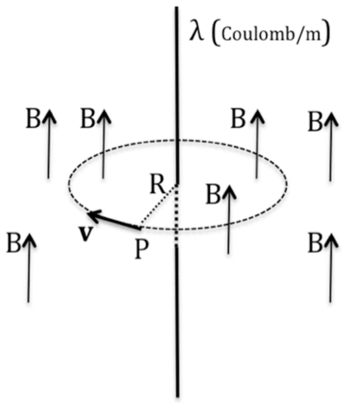


Figure 24.18:

Figure 24.19:

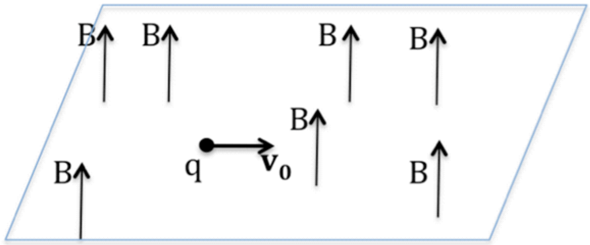


Figure 24.20:

Chapter 25

Bohr's Model of the Hydrogen Atom

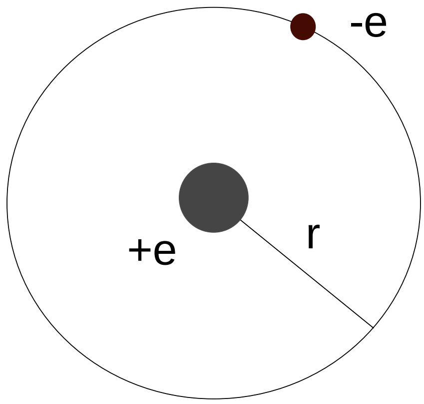


Figure 25.1: Bohr's simple model for the Hydrogen atom

25.1 Classical Physics: Why Are Atoms Like This? How Can Atoms Exist?

The development of Chemistry and Physics in the 18^{th} and 19^{th} centuries provided scientific evidence for reviving the ancient idea of the atom as the fundamental building block of matter. The existence of positive and negative charges together with the observation that matter is usually electrically neutral meant that atoms are also neutral, with equal amounts of positive and negative charge. The particle that carries the negative charge, *the electron*, was discovered in 1895 by the experiments of J.J.Thomson. The distribution of positive and negative charges in atoms was investigated with the experiments of Rutherford, Geiger and Marsden. These experiments yielded the unexpected result that the positive charge was concentrated in a very small *nucleus*, which occupied only 10^{-15} of the atom's volume, while the negatively charged electrons were in the rest of the volume, defining the size of the atom.

An electron is attracted to the nucleus by the electrostatic force, which has a $1/\ r^2$ dependence on the distance r between the electron and the nucleus, just like the gravitational force between the Sun and a planet. In direct analogy with planetary orbits, classical physics pictures electrons to be in orbit around the nucleus of an atom. This is because there must be angular momentum in the atom, otherwise electrons would fall radially into the nucleus, and there would be no charge separation within the atom, contrary to what the Rutherford experiment shows.

The simplest example for studying the atom is the hydrogen atom, which consists of a single proton as its nucleus and a single electron in orbit. According to classical physics the electron's orbit is determined by

$$\frac{e^2}{4\pi\epsilon_0\ r^2} = \frac{mv^2}{r} \tag{25.1}$$

for a circular orbit. This is just Kepler's 3rd Law, obtained from $F = ma$, for the $1/\ r^2$ electrostatic force instead of the gravitational force.

Binary stars bound by the gravitational attractive $1/r^2$ force exhibit no preferred size scale for their orbits. Binary stars are very common: about 70% of the stars observed in the sky are actually binaries. There is no chosen *scale* for the orbit's radius r. Observations show that in nature there are binary star systems of all sizes r. There are *contact* binaries in which the stars touch each other. There are also very widely-separated binaries, in which the binary system almost reaches other neighboring stars. Two different binaries containing stars of similar masses can have very different sizes. Similarly, *planetary systems* like our Solar System contain orbits of many different sizes. In the last two decades, many stars other than our Sun have been found to bind planetary systems. These systems too have varying sizes.

Equation 25.1 by itself implies that just like binary star systems, hydrogen atoms, which are bound systems of a proton and an electron, should also come in all sizes. A proton and an electron should be able to bind together at any radius r, provided that the velocities of the electron and the proton with respect to each other are the correct *Keplerian* velocities for that orbit, as given by Equation 25.1. The potential and kinetic energies on orbit depend on the radius of the orbit. Thus, classical physics also predicts that atoms should exist with all kinds of energies. According to classical physics, since there is no preferred size scale of atoms, there is no preferred energy scale either.

But experiments show that atoms have a size scale. The size of the atoms of any element is of the order of a few ***Angstroms***. The Angstrom is the appropriate length unit for atoms. One Angstrom is $10^{-10}\ m$, or 0.1 *nanometer*.

Together with the size scale, there is also an energy scale of atoms. The *binding* or *ionization* energies, the energies exchanged in many chemical reactions, are of the order of a few Volts per unit charge exchanged. Thus, the typical voltage involved in chemical processes is measured in Volts. This unit was defined, historically, because the first source of electrical energy was the *Voltaic battery* based on controlled chemical reactions. Common batteries still used widely are such voltaic batteries, and their typical voltages are indeed a few Volts. Chemistry in fact investigates reactions in which electrons are exchanged between atoms. The typical energy scale of atoms is the energy of an electron moving through a voltage difference of 1 Volt. This gives the practical energy unit of the atom, the natural energy unit of chemistry and of atomic physics, called the *electron-Volt, eV*. Using the value of the electron's charge, $e = 1.6 \times 10^{-19}C$, the SI value of the electron-Volt is $1eV = 1.6 \times 10^{-19}J$. Classical physics has no explanation for the fact that atoms have a size and an energy scale.

Furthermore, atoms are not even expected to be stable according to classical physics. An electron in orbit is being accelerated by the Coulomb force. An accelerated charge means a time dependent electric current. This sets up time dependent magnetic and electric fields. As we saw in Chapter 24, electromagnetic waves will be radiated away. The atom will lose energy to electromagnetic radiation. Therefore, the atom cannot be stable according to classical physics. If the predictions of classical physics held an atom would change all the time, having less and less energy, and the electron(s) would spiral in, getting closer and closer to the positively charged nucleus, eventually joining the nucleus.

According to classical physics, atoms not only should not have energies and sizes of a certain scale, but worse, no matter what size and energy they were formed with, they could not stay at that configuration. If any atoms were formed, they would rapidly disappear out of existence. Classical physics does not explain the stable existence or the observed properties of atoms!

25.2 Bohr's Model: What Chooses the Size of the Atom?

The choice of a definite size is a familiar property of systems carrying waves. In Experiment 3 of NS101, we observe that waves are excited in a closed system only if an integer number of wavelengths or half wavelengths fit into the size of the system. A familiar example is found in musical instruments. The free length of a guitar string, determined by where the player holds her/his finger, corresponds to one half-wavelength for the note played, and 2, 3, 4,.... half-wavelengths for the harmonics of that note.

Bohr's model of the hydrogen atom introduces an extra postulate, which, together with the classical Equation 25.1, predicts the right size and energy levels for the Hydrogen atom, in surprising agreement with experiment. The content of *Bohr's Postulate* amounts to the very radical statement that the electron behaves as a wave, whose wavelength is given by the *de Broglie* relation:

$$\lambda = \frac{h}{mv}, \qquad [\textbf{de Broglie Wavelength}] \tag{25.2}$$

where h is a fundamental constant of nature called the Planck constant. This constant had already been invoked for the radical explanation of *Max Planck* for *blackbody radiation* and the also radical explanation of the *photoelectric effect* by Einstein[1]. While Bohr's initial statement was about *the quantization of angular momentum*, we shall use an equivalent form of the statement- ***an electron orbit at radius*** r ***is possible only if an integer number of electron wavelengths fits into the circular orbit:***

$$2\pi r = n\lambda = \frac{nh}{mv} \quad [\textbf{Bohr's postulate}\text{: electron waves must fit the atom!}] \tag{25.3}$$

This is a very stringent condition. A full number of wavelengths will fit the orbit for only certain radii. The radii of these special orbits are determined by solving Equations 25.1 and 25.3. One finds that for the orbits allowed by Bohr's Postulate r must have the values:

$$r_n = \frac{n^2\hbar^2}{kme^2} = \frac{(n^2\hbar^2)(4\pi\epsilon_0)}{me^2} \equiv n^2 a_0, \tag{25.4}$$

where $\hbar \equiv h/2\pi$ and

$$a_0 \equiv \frac{\hbar^2}{kme^2} = \frac{\hbar^2\ (4\pi\epsilon_0)}{me^2} = 0.5 \times 10^{-10}\text{m} = 0.5 \text{ Angstrom} \tag{25.5}$$

is the basic length scale (size) of the atom called ***the Bohr radius***. Atoms have these typical

[1]The story of early quantum mechanics, with the contributions of Planck, Einstein, Bohr, de Broglie and others can be found in "Physics for Poets" by March, and "The Scientists" by Gribbin

sizes *as a result of the wave nature of the electron.*

> There is a simulation on NS101 SUCourse on Bohr's Model of the hydrogen atom. When you click on any radius r on the atom model, the program calculates the Keplerian velocity for a circular orbit at that radius according to the classical equation (Equation 25.1), and then the wavelength for that velocity, according to the de Broglie hypothesis, Equation 25.2. The program then makes these waves go round and round on the orbit. For most r the waves overlap to give zero. This is called *destructive interference.* If you choose one of the special orbits marked in red in the simulation, you see that the wave goes around and overlaps with itself, because an exact number of wavelengths fits into the orbit. This is called *constructive interference.* These red orbits are the orbits on which both Equations 25.1 and 25.3 are satisfied.

In the orbit with radius r_n, the corresponding speed is

$$v_n = \frac{ke^2}{n\,\hbar} = \frac{1}{4\pi\epsilon_0}\frac{e^2}{n\,\hbar} \tag{25.6}$$

Substituting these special values r_n and v_n in the expression for the energy,

$$E_n = \frac{1}{2}mv_n^2 - \frac{ke^2}{r_n} = -\frac{1}{2}\frac{ke^2}{n^2 a_o} = -(13.6\ eV)\frac{1}{n^2} \tag{25.7}$$

Thus the energy of the Hydrogen atom cannot have just any value. It can only have the values $-1/n^2$ in fundamental units of 13.6 eV. This unit of energy, the absolute value of lowest possible energy of the H atom, is called the *Rydberg*: 1 Rydberg $\equiv$ 13.6 eV.

> The "−" sign means the electron is *bound* to the proton: it has lower energy than it would have if it were very far from the proton and at rest. The energy of an electron at rest at infinity is defined as zero energy, the bound electron must have energy less than zero.

Finally, let us consider the colours of light and other electromagnetic radiation coming from atoms: frequency (or wavelength) of electromagnetic radiation in the visible band correspond to the *colour* of light as our eyes perceive it. 19^{th} century scientists had discovered that when light from atoms, for example in a hydrogen gas, is separated into its colours with a prism or spectral grating, only very special frequencies are found to be present. It was known that electromagnetic waves from hydrogen atoms come only in special frequencies f obeying the formula:

$$f = \text{constant}\left(\frac{1}{n^2} - \frac{1}{n'^2}\right), \qquad \text{where } n \text{ and } n' \text{ are integers!} \tag{25.8}$$

Within classical physics, there is no way to understand this simple but very special formula. This requires another result from quantum mechanics, the *Planck Hypothesis*, historically the first instance of wave-particle duality which Planck invoked to explain succesfully the spectrum of *blackbody radiation.* According to the Planck Hypothesis ***electromagnetic***

waves travel in packets (quanta) of energy called "photons". The energy ε of the photon is proportional to the frequency of the electromagnetic wave:

$$\boxed{\varepsilon = hf = \hbar\omega \qquad [\textbf{Photon Energy}] \qquad (25.9)}$$

The hydrogen atom can only have certain values of energy. The energy ε of a photon emitted by the hydrogen atom must be the energy of the atom in the initial state before emitting the photon minus the energy of the atom in the final state. Using Equations 25.7 and 25.9, one can obtain the experimentally observed result (Equation 25.8). The coefficient in Equation 25.8 can be calculated from the Bohr's model. The model gives the same value for the coefficient as determined empirically from observed spectra: $-13.6\, eV/h$.

The empirically determination of the value of the spectroscopic constant in Equation 25.8 is one of many ways of determining Planck's constant h from experiments. Early determinations of h from totally unrelated experiments like the spectrum of blackbody radiation, the photoelectric effect and spectroscopy of gaseous hydrogen all gave the same value. This was a triumph for the ideas of *wave-particle duality*, and clearly established Planck's constant as a fundamental constant of Nature. The value of h turns out to be $h = 6.63 \times 10^{-34}$J-s.

Solved Problem: Bohr's Model

Consider a particle of mass m in a circular orbit of radius r. The potential energy of the particle is $U(r) = Cr^4$, where C is a constant.

(a) Find the force and write the equation of motion for the circular orbit of the particle.

$\mathbf{F} = -\frac{dU}{dr}\hat{\mathbf{r}} = -4Cr^3\,\hat{\mathbf{r}}$, so the equation of motion ($\mathbf{F} = m\mathbf{a}$) is:

$$4Cr^3 = m\frac{v^2}{r}$$

The minus sign was dropped because both **F** and **a** are in the same direction (towards the center of the circular orbit).

(b) Find the possible orbital radii r_n according to the Bohr model.

To find r_n, we need to use the equation of motion above, along with the Bohr's postulate (Equation 25.3); $2\pi r = nh/mv$. Solving these two equations together for r_n, we find:

$$4Cr^3 = \frac{mv^2}{r} = \frac{m}{r}\left(\frac{n^2h^2}{m^2 4\pi^2 r^2}\right)$$

$$\text{so, } r_n = \left(\frac{n^2\hbar^2}{4Cm}\right)^{1/6}, \text{ where } \hbar = h/2\pi$$

(c) Find the possible energy levels for this electron according to the Bohr model.

$E = \frac{1}{2}mv^2 + U(r)$, and we know v_n and r_n from (a) and (b) above, so

$$\begin{aligned} E_n &= \frac{1}{2}mv_n^2 + Cr_n^4 \\ &= \frac{m}{2}\left(\frac{n^2h^2}{m^2 4\pi^2 r_n^2}\right) + C\left(\frac{n^2\hbar^2}{4Cm}\right)^{2/3} \end{aligned}$$

This simplifies to

$$= \frac{3}{4}\left(\frac{4Cn^4\hbar^4}{m^2}\right)^{1/3}$$

CHAPTER 25 - PROBLEMS:

1. **Virial Theorem:** Using the equation of motion, Equation 25.1, show that the electrostatic potential energy is -2 times the kinetic energy, $U = -2(KE)$, as used in Equation 25.7.

2. (a) The power P radiated by an accelerated charge q with acceleration a is given by the formula

$$P = \frac{1}{6\pi\epsilon_0}\frac{q^2a^2}{c^3} \text{ Watts}$$

where c is the velocity of light. Estimate how long it will take for the electron in a Hydrogen atom to spiral in from an orbit of radius r to an orbit of radius $r/2$

according to classical physics? Take $r \simeq a_0$.

(b) Estimate the classical decay time for the atom to decay from the orbit with radius $r_2 = 4a_0$, with energy E_2 to the orbit with radius $r_1 = a_0$, with energy and energy E_1. A correct quantum mechanical calculation for the decay time of a free hydrogen atom gives a similar result, and this agrees also with experimental results.

(c) Estimate the classical decay time for the atom to decay from the lowest energy orbit with radius $r_1 = a_0$, and energy E_1, until the electron spirals into the nucleus, at $r_N = 10^{-14}m$. According to quantum mechanics, as borne out by experiments, such decay *does not occur at all.* The lowest energy state is stable, because it is not possible for an electron-wave to exist at any lower energy, in an orbit closer to the nucleus than $r = a_0$. The wave fill not fit into any smaller space – this is the *Uncertainty Principle*, to be discussed in the Chapter 27.

3. Derive the results of Equations 25.4, 25.6 and 25.7 from Equations 25.1 and 25.3.

4. Derive Equation 25.8 and derive the constant in terms of e, m and $\hbar$.

5. Calculate the orbital velocity v_n of the electron in the n^{th} Bohr orbit, for $n = 1, 2, 3$.

6. Although the electron in the hydrogen atom is non-relativistic, it is interesting to write its energy scale E_1 and radius scale, the Bohr radius a_0 in terms of the dimensionless *Fine Structure Constant* α, which contains the speed of light c,

$$\alpha \equiv \frac{1}{4\pi\epsilon_0}\frac{e^2}{\hbar c} \cong \frac{1}{137}$$

(a) Express the ground state energy E_1 of the hydrogen atom in terms of the fine structure constant α and the rest mass energy $E_0 \equiv m_e c^2 \cong 511keV$ of the electron.

(b) Express the Bohr radius a_0 in terms of α and the electron's "*Compton wavelength*" $\lambda_e \equiv \hbar \,/\, (m_e\, c)$.

7. (a) What is the frequency f and radian frequency ω of the radiation (photon) emitted when a hydrogen atom decays from its quantum state n with energy E_n to the quantum state $n-1$ with energy E_{n-1}?

(b) What is the radian frequency of the emitted radiation (photon) in the limit of large n, in terms of e, r_n, n and $\hbar$?

(c) Classically, an electron in a circular orbit of radius r_n around the proton emits electromagnetic radiation whose radian frequency is the same as the angular frequency of the electron in its orbit. Calculate this frequency ω_n for a classical orbit whose radius is the n^{th} Bohr radius r_n. You will find that this frequency agrees with the photon frequency calculated above in the limit of large n. This is an instance of the *Correspondence Principle*, which state that quantum mechanical results for situations with small values of the energy quanta agree with classical results.

Chapter 26

Wave-Particle Duality

From the end of the 19^{th} century, the deficiency of classical physics in explaining fundamental experiments led to the proposal of very radical hypotheses. These new hypotheses did explain the experiments astonishingly well and made predictions that were subsequently verified. The various quantum hypotheses had their successes at the expense of the *dichotomy between waves and particles* in classical physics. Before the quantum revolution, physical entities in Nature were *either* particles *or* waves.

Particles were objects with mass and possibly charge, moving on trajectories determined by Newton's Second Law. The word "particle" suggests microscopic objects, basic building blocks of matter like atoms and their constituents, such as electrons, nuclei, protons and neutrons. One can extend the sense of the word "particle(s)" to include bodies with mass, made of macroscopic numbers of particles. Such macroscopic systems, whether they are rigid bodies or fluids like a cloud, also obey the laws of classical mechanics, rigid body dynamics, or fluid dynamics, all derived from Newton's Second Law.

Waves in classical physics are *collective motions* within systems made of very large numbers of particles, like *surface waves* in water or *sound waves*, which are traveling oscillations of density and pressure in air, or in any fluid or solid macroscopic system. Such waves are in one sense not fundamental entities the way atoms and their constituents are, because they occur in systems composed of large numbers of atoms. In another sense, waves (or *normal modes*) of macroscopic systems can be considered as the fundamental entities making up the dynamics of these systems because they describe the macroscopic system as *continua* and describe all the possible macroscopic motions within the system.

With the understanding of electromagnetic fields and waves in the 19^{th} century, namely waves which could travel in vacuum, the question arose whether vacuum, devoid of all material particles, nevertheless contained a medium without mass called the *ether*, which was supposed to be the continuum in which electromagnetic fields and waves existed. The idea that any wave had to travel in a medium was only a suggestion by analogy with sound waves, and other waves known to exist in material media. When experiments failed to detect any measurable consequences of the ether, the search was abandoned. Electromagnetic fields and waves exist in vacuum without any medium to support them. Thus in classical physics, electromagnetic waves are fundamentally, microscopically, waves.

From the Scientific Renaissance on, Light, the subject of the science of Optics, was proposed to be a particle on the basis of some experiments, and a wave on the basis of other experiments, which showed that light exhibited *interference* and *diffraction*. The notion that it could be *both* wave and particle was totally foreign to classical physics, so there were heated debates on *whether* light was *really* made of particles or whether it was *really* a kind of wave. At the end of the 19^{th} century, the issue seemed to be closed with the realization that light was an electromagnetic wave. But soon afterwards, the first radical proposals for wave-particle duality were made for light and other electromagnetic waves.

First came Planck's Hypothesis, in the year 1900, that electromagnetic waves carried energy in particle-like *quanta*. Planck used this hypothesis to provide a successful explanation of the observed *spectrum of blackbody radiation*, the electromagnetic radiation emitted by any object in thermodynamic equilibrium at some temperature T. In 1905, Einstein used Planck's hypothesis to explain the *photoelectric effect*, which is the observation that the energy of the electrons emitted by a solid on which light (or some other electromagnetic wave) is shined, increases with the frequency of the electromagnetic wave. These first two instances of the hypothesis of wave-particle duality stated that light, which was classically thought to be a wave, had a particle nature *also*. With Bohr's model of hydrogen atoms came the proposal that electrons, which were thought to be particles classically, were *also* waves.

26.1 Particle Properties

The energy of a classical particle depends on its momentum $\mathbf{p}$ and position $\mathbf{r}(x, y, z)$, so $E = E(\mathbf{p}, \mathbf{r})$.

For every different type of particle and every physical situation, the particular expression for E is different. For a non-relativistic particle (non-relativistic means "moving at speeds much less than c"), it is:

$$E = \frac{p^2}{2m} + U(x, y, z). \tag{26.1}$$

This expression is different for particles of different mass m and for different potential energies $U(x, y, z)$. Such a relation between energy and momentum and position describes the physics of a particle under specific conditions.

For a relativistic particle, moving at a speed comparable to c, the energy is:

$$E = \sqrt{m^2c^4 + p^2c^2} + U(x, y, z) \tag{26.2}$$

26.2 Wave Properties

There are many different types of waves that are described by different wave equations. For each wave system, there is a particular relation between the frequency and wavelength. First, let us remember the various wave properties:

- T: *the period*, the time for the wave to return to the same state at a given point.

- f : *the frequency*, the change in the phase of the wave (in radians) with respect to time (seconds). $f = 1/T$. Unit: $s^{-1} \equiv Hertz, Hz$.

- $\omega = 2\pi f = 2\pi/T$: *the angular or radian frequency* : each period or cycle contains 2π radians of phase. Unit: radians / second, $rad\ s^{-1}$.

ω appears as the coefficient of t in the expression for the wave. For example the wave might be described by a function:

$$\psi(x,t) = \sin(kx - \omega t)$$

- λ : *the wavelength*, at any time, the distance interval at which the wave repeats itself in space. SI unit: meters, m.

- $k = 2\pi/\lambda$: *the wave number*, the change in the phase of the wave (in radians) per unit distance. SI unit: radians / meter, $rad\ m^{-1}$.
  k appears as the coefficient of x in the expression for the wave, such as $\psi(x,t) = \sin(kx - \omega t)$.

The wave function $\psi(x,t)$, or, more generally in three dimensions, $\psi(x,y,z,t)$, has different meanings in different physical systems. For waves on the surface of water, ψ means the positive or negative difference (height) of the surface of the water with respect to a flat surface. For sound waves ψ is the change in the pressure or density of the medium as the wave passes. For electromagnetic waves, ψ may be the electric field or the magnetic field.

The physics of any wave system is described by a relation between the frequency and wavelength of the wave:

$$\omega = v(k)\ k, \tag{26.3}$$

or, equivalently, as you can see by substitution,

$$\lambda/T = \lambda f = v(k) = v(\lambda). \tag{26.4}$$

Such a relation, called the *"dispersion relation"*, exists for all waves. It describes the physics of the wave, just as the relation $E = E(\mathbf{p}, \mathbf{r})$ describes the physics of a particle. The dispersion relation comes from the wave equation for that particular wave, so it comes from the basic physics of that wave, which may include $F = ma$ and the gas laws for sound, or the Maxwell Equations for the electromagnetic waves. The quantity $v(k)$, called the ***phase velocity***, in general depends on the wavenumber k (equivalently on wavelength λ). In some cases the phase velocity is the same for all wavelengths.

For example, for waves on a wire with tension τ and mass per unit length ρ, the phase velocity is $v = \omega\ /\ k = \lambda\ f = \sqrt{\tau/\rho}$. For electromagnetic waves in vacuum, $v = \omega\ /\ k = \lambda\ f = c$. The speed of light in vacuum, c , does not depend on the wavelength. But for electromagnetic waves in glass, say light traveling in a prism, the velocity does depend on the wavelength (on color!), so $\omega = v(k)k$. White light going through a glass prism emerges broken up into the different colors in its spectrum because the different wavelengths (colors) propagate at different velocities in glass. The function $v(k)$ is different for different materials.

A *wave packet* is the superposition of many waves of different wavenumbers k and frequencies $\omega(k)$. When many waves are superposed, the wave packet will also move, since all of the constituent waves of different k and $\omega(k)$ move. The velocity of the wave packet as a whole is *not* the phase velocity $v(k) = \omega\ /\ k$ of any of its constituents. Rather, the wave packet moves with the ***group velocity***:

$$v_g(k_0) \equiv \frac{\partial \omega}{\partial k}(k_0),$$

the partial derivative of the angular frequency $\omega(k)$ with respect to the wavenumber k, evaluated at the average wavenumber k_0 of the band of wavenumbers k making up the wave packet. (We shall not derive this relation here).

We shall discuss the wave packets and the *Uncertainty Relation* in the next Chapter.

26.3 Dictionary of Wave and Particle Properties

To understand basic properties of matter, structure of atoms, molecules and bulk matter, and also of radiation, one has to take into account the dual nature of all things, that particles are at the same time waves, and waves are also particles. Particle properties $E = E(\mathbf{p}, \mathbf{r})$ and wave properties $\omega = v(k)\ k$ are specific to each particular system. *But the translation between wave and particle properties is universal. There are two entries to this dictionary*, which applies to electrons, protons, neutrons, all other elementary particles, to atoms and molecules, large and small, to electromagnetic radiation (*photons*), to sound (*phonons*) and so on:

$$\boxed{\begin{aligned} E &= hf = \hbar\omega && \textit{the Planck relation} \\ p &= h/\lambda = \hbar k && \textit{the de Broglie relation} \end{aligned}}$$

The universal constant h, the Planck constant, occurs in both entries of the wave-particle dictionary, which applies to all systems in Nature. Its value and the value of the frequently used combination $\hbar = h/2\pi$ in SI units are:

$$\begin{aligned} h &= 6.6260755 \times 10^{-34}\ \textit{J-s}, \text{ and} \\ \hbar &= \frac{h}{2\pi} = 1.05457266 \times 10^{-34}\ \textit{J-s}. \end{aligned}$$

Let us use the dictionary for electromagnetic radiation, for which $\omega = ck$:

Multiplying both sides of $\omega = ck$ with $\hbar$ gives $E = pc$. The particles of electromagnetic radiation must have this relation between their energy and momentum. Comparing with the relativistic energy momentum relation for free particles, $E = \sqrt{m^2c^4 + p^2c^2}$ (from Equation 26.2), one sees that the photons are massless particles, namely $m = 0$.

Now use the dictionary for a free, non-relativistic electron, an electron in empty space so the potential energy is zero. $E = p^2/2m$ gives, substituting from the dictionary,

$$E = \frac{\hbar^2 k^2}{2m} = \hbar\omega, \quad \text{so} \quad \omega(k) = \frac{\hbar k^2}{2m} \tag{26.5}$$

for the angular frequency $\omega(k)$ of a free non-relativistic electron of wave number k. So the phase velocity of the electron wave is

$$v(k) = \frac{\omega}{k} = \frac{\hbar k}{2m} = \frac{p}{2m}. \tag{26.6}$$

The particle-like nature, finite size, of the electron is represented by a wave packet. The group velocity of the wave packet made of many electron waves is

$$v_g(k_0) = \left(\frac{\partial \omega}{\partial k}\right)(k_0) = \frac{\hbar k_0}{m} = \frac{p_0}{m}, \tag{26.7}$$

using the free electron dispersion relation, Equation 26.5. Both the phase velocity and the group velocity of the electron waves or wave packet are *wavelength (wave number) dependent. It is the group velocity of the wave packet that agrees with the particle velocity, p_0/m.*

26.4 What is the Meaning of the Particle Wave?

In quantum mechanics, the wave function associated with a particle has the meaning that *the probability of the particle being near the point x, y, z in a small volume $dxdydz$ is equal to $|\psi(x, y, z, t)|^2 dxdydz$.* This *probabilistic* interpretation of the particle wave, due to Max Born[1], is very counter-intuitive, in terms of our intuitions shaped by the macroscopic world where particles are particles and waves are waves.

Let us return to the quantum world and ask: where is the electron? The answer is, it is here and there and everywhere, spread out to be here or there with various probabilities. For a particle, this is a very strange property. One can in principle make an experiment to find out where the particle is. In practice, too, one can make such experiments, to find out where the particle is to an accuracy of $\Delta x, \Delta y, \Delta z$. With developing technology one actually can make such experiments to locate a particle, say an electron, within a volume $\Delta x \Delta y \Delta z$ around a particular position, (x_0, y_0, z_0). After all, the electron or any other particle, *is a particle and a wave.* As a particle, in those experiments designed to find out where it is, the electron should be somewhere, and it turns out that it is indeed found to be somewhere.

Now consider the electron in the hydrogen atom. Nobody is making an experiment to find out where the electron is. "Nobody" does not just mean no human beings: it means that, most of the time, interactions of other atoms, photons, or "things" are not disturbing the atom, either to squeeze its electron into some small volume, or to change its state. At other times there are interactions of the hydrogen atom. Consider the isolated hydrogen atom, in one of its possible energy states, with energy E_n, not interacting with anything. Where is the electron? It is spread out around the atom, described by a wave function. The *average* distance of the electron from the proton in this spread out wave of the state with energy E_n is just $r_n = n^2 a_0$.

Born's probabilistic interpretation of the quantum mechanical wave is supported by an analogy with the electromagnetic waves, which are also particles, *photons.* As we saw in Chapter 25, electromagnetic fields carry an energy per unit volume of space, given by

$$u_{EM} = u_B + u_E = \epsilon_0 \, \mathbf{E}^2(x, y, z, t) = \frac{1}{\mu_0} \, \mathbf{B}^2(x, y, z, t). \tag{26.8}$$

Maxwell's Equations show that the total energy of the electric and magnetic fields in an electromagnetic wave can be expressed in terms of either the electric field or the magnetic field. One can express the electromagnetic energy density in terms of either field. With the Planck Hypothesis, the energy density in the waves of frequency f is carried as photons,

[1]see, e.g., `http://www-history.mcs.st-andrews.ac.uk/Biographies/Born.html`

which are the particles corresponding to electromagnetic radiation. Since the energy of each photon is $\varepsilon = hf$, the number n_f of photons of frequency f, per unit volume, near the point (x, y, z), is

$$n_f(x, y, z, t) = \frac{1}{hf}\ u_{EM;f}(x, y, z, t) = \frac{1}{hf}\ \epsilon_0\ \mathbf{E}_f^2(x, y, z, t), \tag{26.9}$$

where the subscript f denotes waves of the frequency f. The number of photons in a small volume $dx\ dy\ dz$ around the point (x, y, z) is:

$$n_f(x, y, z, t)\ dx\ dy\ dz = \frac{1}{hf}\ \epsilon_0\ \mathbf{E}_f^2(x, y, z, t)\ dx\ dy\ dz \tag{26.10}$$

If there is a total number of N_f photons of frequency f in a system, the *probability* of finding photons of frequency f in the volume $dx\ dy\ dz$ near the point (x, y, z) is

$$P_f(x, y, z, t)\ dx\ dy\ dz \equiv \frac{n_f(x, y, z, t)\ dx\ dy\ dz}{N_f} = \frac{\epsilon_0\ \mathbf{E}_f^2(x, y, z, t)\ dx\ dy\ dz}{hfN_f}. \tag{26.11}$$

The probability $P_f(x, y, z, t)$ per unit volume at (x, y, z) of finding the particle there is indeed proportional to the absolute-value-squared of the wave function $(\mathbf{E}^2(x, y, z, t))$, for electromagnetic waves-photons. Since wave-particle duality is confirmed by experiment to be a property of all things, the wave functions of electrons and other particles should also have this meaning, that *the absolute value squared of the wave function at* (x, y, z) *gives the probability density, the probability per unit volume, of finding the associated particle in a small volume near* (x, y, z). This is the Born interpretation of the wave function.

26.5 The Schrödinger Equation

Wave functions describe the distribution of some wave amplitude, some physical quantity, in space and time. The wave function in each system obeys a wave equation that reflects the laws of physics governing the system. The wave function is obtained from the wave equation.

We saw in Chapter 24 that the wave functions for the electric and magnetic fields in an electromagnetic wave are obtained from a *wave equation.* The wave equation itself is derived from the Maxwell Equations, which contain the complete information about electric and magnetic fields, as derived from experiments.

For waves running on a stretched wire, the wave function describes the displacement of each piece of the wire from its straight equilibrium configuration. The wave equation is derived from Newton's 2^{nd} Law for each piece of the wire, moving under the influence of forces between that piece and neighboring pieces.

For sound waves in air, in a fluid or solid object, the wave function describes the changes of density and pressure (or stresses) at each location, compared to the equilibrium situation of uniform density and pressure everywhere. The wave equation is again just Newton's 2^{nd} Law for each small volume element of the system, moving under the influence of forces between that volume element and neighboring elements.

Likewise, to determine the wave function that gives the probability of a particle being here or there, one needs a wave equation. The Bohr model does not give any detailed information on the wave function. By capturing the crucial essence of the wave (that it fits the atom, so to speak) the Bohr model derives the energy and radius values of the hydrogen atom, and the spectrum of photons emitted and absorbed by the atom correctly and astonishingly, given that the model combines a classical equation with the Bohr condition on a mysterious wave. But where is the electron? It cannot be on a classical orbit as a point particle – that is not even stable according to classical physics. The Bohr radius cannot be the radius of an orbit. It is some average position of the electron wave, according to the Born interpretation of the particle wave function. One needs a wave equation to calculate the wave function and to derive the properties of the atom from the wave function.

To derive the wave equation, we start with the assumption that the true wave function is a wave packet that is a superposition of cosine and sine functions. Consider a component of the wave function of the form,

$$\psi_1(x) = A\cos(kx). \tag{26.12}$$

The characteristic property of a cosine or sine function is that its second derivative is a negative constant times the same function:

$$\frac{d^2\psi(x)}{dx^2} = -k^2\ \psi(x). \tag{26.13}$$

From the wave-particle dictionary, the momentum is $p = \hbar k$, and the kinetic energy for a non-relativistic particle is

$$E_K = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m}.$$

For a free particle, total energy is equal to the kinetic energy, $E = E_K = \hbar^2 k^2/2\mathrm{m}$, Equation 26.13 implies that the wave function $\psi(x)$ must satisfy the wave equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_1(x)}{dx^2} = E_K\ \psi_1(x). \tag{26.14}$$

A simple three dimensional wave function to generalize all this is

$$\psi_3(x, y, z) = A\cos(k_x x\ +\ k_y y\ +\ k_z z). \tag{26.15}$$

The *wave vector* $\mathbf{k} \equiv (k_x, k_y, k_z)$ is a vector whose magnitude is related to the wavelength λ with the standard relation $k = 2\pi\ /\ \lambda$, and whose direction gives the direction of travel of the time-dependent form of the wave function,

$$\psi_3(x, y, z, t) = A\cos(k_x x\ +\ k_y y\ +\ k_z z - \omega\ t).$$

In three dimensions the de Broglie relation in the wave-particle dictionary has the form,

$$\mathbf{p} = \hbar\mathbf{k}. \tag{26.16}$$

It follows that the three dimensional wave function $\psi_3(x, y, z)$ to represent a free non-relativistic particle of mass m satisfies the wave equation

$$-\frac{\hbar^2}{2m}\left[\frac{\partial^2\psi_3}{\partial x^2}\ +\ \frac{\partial^2\psi_3}{\partial y^2}\ +\ \frac{\partial^2\psi_3}{\partial z^2}\right] = E_K\ \psi_3. \tag{26.17}$$

Now, this is a linear equation. So *any superposition* of cosine or sine waves, each of which satisfies Equation 26.17 must also be a solution of this wave equation. *All wave packets* *describing a free non-relativistic particle of mass m are described by wave functions that are solutions of the wave equation (Equation 26.17).*

If the particle is not free, but is interacting with other objects, with conservative forces described by a potential energy $U(x, y, z)$, the total energy $E = E_K + U(x, y, z)$ must be a conserved quantity (i.e., a constant). Therefore, from the identity:

$$E_K\ \psi + U(x, y, z)\ \psi = E\ \psi, \tag{26.18}$$

which any wave function $\psi = \psi(x, y, z)$ describing the interacting, non-relativistic particle must satisfy, we obtain

$$-\frac{\hbar^2}{2m}\left[\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}\right] + U(x, y, z)\ \psi = E\ \psi, \tag{26.19}$$

[Schrödinger Equation]

using, as we did for the free particle, the de Broglie relation to express $E_K\ \psi(x, y, z)$ in terms of second derivatives of the wave function $\psi(x, y, z)$. This is the ***time-independent Schrödinger Wave Equation***[2] for a non-relativistic particle of mass m and potential energy function $U(x, y, z)$[3]. One must solve this wave equation, with the potential energy and the *boundary conditions* of a given situation.

For the electron in the hydrogen atom, the electrostatic potential energy is

$$U(x, y, z) = -\frac{1}{4\pi\epsilon_0}\frac{e^2}{r} = -\frac{1}{4\pi\epsilon_0}\frac{e^2}{(x^2 + y^2 + z^2)^{1/2}} \tag{26.20}$$

and the Schrödinger Equation is

$$-\frac{\hbar^2}{2m}\left[\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}\right] - \frac{1}{4\pi\epsilon_0}\frac{e^2}{(x^2 + y^2 + z^2)^{1/2}}\psi = E\ \psi(x, y, z). \tag{26.21}$$

This is a quite special, non-trivial *differential equation*. It turns out that there are solutions for negative energy values, that is, for the electron to be bound to the proton, only for the discrete quantized values

$$E_n = -\frac{1}{4\pi\epsilon_0}\frac{e^2}{2n^2 a_o} = -(13.6\ eV)\frac{1}{n^2}. \tag{26.22}$$

These are the same values that are correctly given by the Bohr Model. For each of these possible energy values E_n a special wave function $\psi_n(x, y, z)$ is obtained as a solution of the Schrödinger Equation. We will not solve the Schrödinger Equation for the hydrogen atom here; you will learn how to do that in Quantum Mechanics courses. The square of the absolute value of this wave function $\psi_n(x, y, z)$ gives the probabilities of finding the electron here or there, at a small volume $dxdydz$ near any point (x, y, z), when the atom is in the

[2]The *time dependent* form of the Schrödinger Equation can be obtained by similar arguments. We will not discuss the Time Dependent Schrödinger Equation here as it unavoidably involves *complex numbers*.

[3]In using the Schrödinger Equation, one need not worry whether there is a potential energy to describe the system, or whether there are any non-conservative forces: in microscopic systems, all forces are conservative forces, derived from the electrostatic force in the case of atoms.

state with energy E_n:

$$P_n(x, y, z)\ dxdydz = |\psi_n(x, y, z)|^2\ dxdydz. \tag{26.23}$$

This probability distribution contains *all* the possible information one can calculate, or check experimentally, about the properties of the atom. For example, the average distance r_n of the electron from the proton, when the atom is in the state with energy E_n, is calculated as a *weighted average* using the probability distribution $P_n(x, y, z)$. The answer is exactly the value given by the Bohr Model, $r_n = n^2 a_0$!

"Particle in a Box"

Consider a particle confined to move only along a line of length L, with coordinate $0 < x < L$. Inside the box, the particle is free, the potential energy $U(x) = 0$ for $0 < x < L$ while outside the box the potential energy is very large, "infinite", so the particle does not have enough energy to get out of the box. This is a realistic approximation for some important physical situations.

For example, the particle might be an electron moving almost freely, with approximately zero force on it, with approximately constant potential energy, which we take to be $U(x) = 0$, inside a straight linear polymer chain of length L, while being strongly bound to the polymer chain, so it cannot leave the molecules' length, the region $0 < x < L$. Another situation where the particle-in-a-box model is useful is for electrons confined in a film of thickness L, sandwiched between different materials, the electron being free to move within the film but not having enough energy to move into the different materials at $x < 0$ or $x > L$.

Solved Problem: Particle in a Box

Consider a particle in a box problem where a particle is confined to move in a one-dimensional box of size L. Inside the box $(0 < x < L)$ the particle is free, and outside the box the potential energy is infinite. Follow the steps to find the structure of the wave function.

(a) What is the Schrödinger Equation for a free particle, of mass m, valid within the "box", $0 < x < L$?

> Following Equation 26.17, Schrödinger Equation for a free particle in one dimension is:
>
> $$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} = E_k\ \psi(x),$$
>
> where $\hbar$ is the Planck's constant, m is the mass of the particle and E_k is the energy of the particle. For a free particle, E_k is the kinetic energy of the particle since there is no potential energy.

(b) The solutions of the Schrödinger Equation for a free particle are of the form $\psi(x) = A\ \cos(kx)$ or $\psi(x) = B\ \sin(kx)$, where A and B are constants. Since the probability for the particle to be at $x \leq 0$ must be zero, which of these wave functions is the correct form for the boundary conditions of the box?

> Since the wave function should vanish at $x = 0$, $\psi(0) = 0$, and the solution should be of the form $\psi(x) = B\ \sin(kx)$.

(c) Since the probability for the particle to be at $x \geq L$ must also be zero, what are the possible values of k to satisfy this boundary condition?

> Since the wave function should also vanish at $x = L$, $\psi(L) = 0$, $\sin(kL) = 0$. This is possible only if $kL = n\pi$, where n is an integer $n = 1, 2, ...$. So the possible values of k are
>
> $$k = n\pi/L.$$
>
> In other words the wavenumber can only have discrete values determined by the size of the box.
> So, the wave function can be written as $\psi(x) = B\sin(\frac{n\pi}{L}x)$.

(d) What are the possible values of the particle's wavelength?

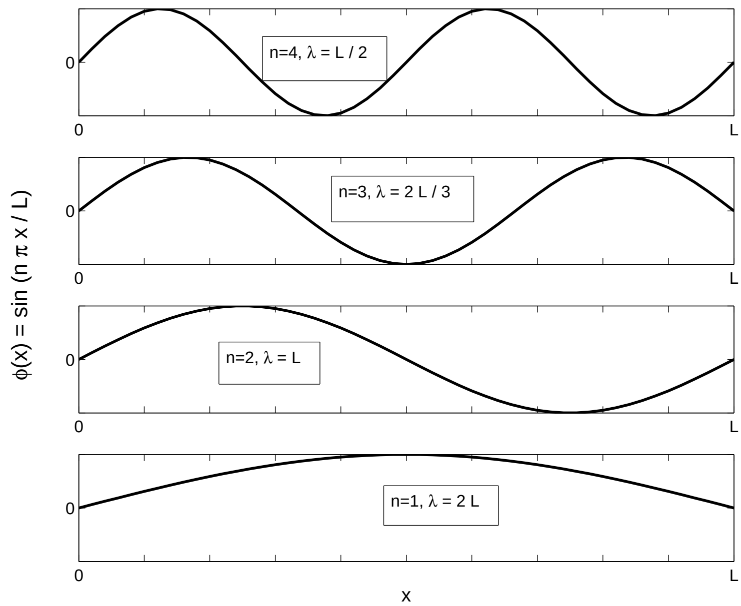
> Since the wavelength and the wavenumber are related as $\lambda = 2\pi/k$, the possible values of the particle's wavelength can be obtained by substituting the possible values of k obtained in the previous question:
>
> $$\lambda = 2L/n$$

Continues on next page...

...Continued from previous page

The wavelength also can have only discrete values, which are determined by the size of the box.

The figure below shows the wave functions $\psi(x)$ inside the box, for 4 different λ values, corresponding $n = 1, 2, 3$, and 4.



26.6 Fundamental Constants, Fundamental Forces and Wave-Particle Duality

Together with Newton's Universal Gravitational Constant G and the Speed of Light c, the Planck constant, h is one of the fundamental constants of Nature. The dimensionless *Fine Structure Constant*,

$$\alpha \equiv \frac{e^2}{4\pi\epsilon_0\, \hbar\, c}$$

means that a unit of charge can be constructed dimensionally from h and c. As we saw in Chapter 4, fundamental mass, length and time units, the Planck units, can be defined in terms of the three constants, G, c and h.

A quantum theory of electromagnetism, *Quantum Electrodynamics*, making full use of the wave particle duality in the treatment of both particles and fields, has been successfully developed in the 1940s (Feynman, Tomonaga and Schwinger, Nobel Prize in Physics, 1965), and checked with experiments. The dimensionless Fine Structure Constant has a key role in the expression of Quantum Electrodynamics. The Strong and Weak Nuclear Forces were unified with Electromagnetism in a set of Grand Unified Quantum Theories (Weinberg, Salam, Glashow, Nobel Prize in Physics, 1979).

Of the four fundamental forces of Nature, gravity still remains not quantized. The phenomena of gravitation, which became fully understood in the classical context in Einstein's

General Theory of Relativity must also have quantum aspects. The Gravitational Field has wave solutions propagating in vacuum with velocity c. This is confirmed indirectly by the behavior of close binaries of neutron stars, whose dynamics indicate that they must be emitting gravitational waves (Hulse and Taylor, Nobel Prize in Physics 1993). High technology antenna and interferometer systems are now under construction for direct observation of gravitational waves arriving here on Earth. But wave particle duality in connection with gravitational fields has not been understood yet. The Planck mass, length and time scales must have played a significant role in the earliest epochs of our Universe following the Big Bang. The actual physical meaning of the Planck scales will be clear when a valid theory of quantum gravity is developed and confirmed by experiments. The full significance of the Planck constant h will be realized only with an understanding of quantum gravity.

CHAPTER 26 - PROBLEMS:

1. Express the velocity v for a free electron (potential energy $U = 0$) in terms of its wavelength λ and mass m.

2. Nonrelativistic free electrons: What is the momentum and wavelength of an electron moving with velocity
   (a) $v = 1\ m/s$?
   (b) $v = 10^3\ m/s$?
   (c) $v = 10^6\ m/s$?

3. A wave packet for a free nonrelativistic electron contains wavelengths in the band from 0.5π Angstroms to π Angstroms,
   (a) What is the range of wavenumbers k in this wave packet?
   (b) The average wavenumber for this wave packet is $k_0 = 3 \times 10^{10}\ m^{-1}$. What is the phase velocity of the wave at this wavenumber?
   (c) What is the group velocity of the wave packet?
   (d) What is the momentum of the electron represented by this wave packet?

4. Relativistic free electrons: What is the momentum and wavelength of an electron moving with total energy E?
   (a) For $E = 1\ MeV$?
   (b) For $E = 10^3\ MeV$?
   (c) For $E = 10^6\ MeV$?

   Compare the electron wavelengths with atomic and nuclear dimensions, 1 *Angstrom* and 10 *Fermi* (1 Fermi=10^{-15} m, 1 Angstrom = $10^{-10}\ m$)

5. What is your wavelength when

(a) you are walking at a speed of 2 m/s ?

(b) you are running at a speed of 10 m/s ?

(c) you are at rest ? (We will see in the next Chapter that you *cannot* be at rest).

6. What is the total energy of the free nonrelativistic electron

   (a) in terms of its momentum p and its mass m?

   (b) in terms of its wavenumber k?

   (c) in terms of its frequency?

7. What is the total energy of the free relativistic electron

   (a) in terms of its momentum p and its mass m?

   (b) in terms of its wavenumber k?

   (c) in terms of its frequency?

8. **The "Particle in a Box":** Previously in a solved problem, we have found the structure of the wave function for a free particle confined in a box with infinitely hard walls and of size L. The particle is freely moving inside the box ($U(x) = 0$), while outside the box the potential energy is"infinite", and the particle does not have enough energy to get out of the box.

   (a) What are the possible values of the particle's momentum?

   (b) What are the possible values of the particle's energy? (The energy is only kinetic energy, since the particle is free inside the box).

   (c) What is the minimum possible energy of the particle in the box? Can the particle be at rest, with $E = E_k = 0$?

   (d) Estimate the minimum energy of an electron moving freely in a linear molecule of length 100 *Angstroms*. What is the wavelength and the momentum of the electron in this minimum energy state?

   (e) A runner of mass 50 kg is on a 400 m track. She wants to relax before the race, but according to quantum mechanics she cannot be at rest. What is the minimum energy that this runner can have? What is her wavelength and her momentum in this state of minimum energy?

Chapter 27

Wave Packets and the Uncertainty Relation

27.1 Wave Packets

What happens when many cosine (or sine) waves are superposed?

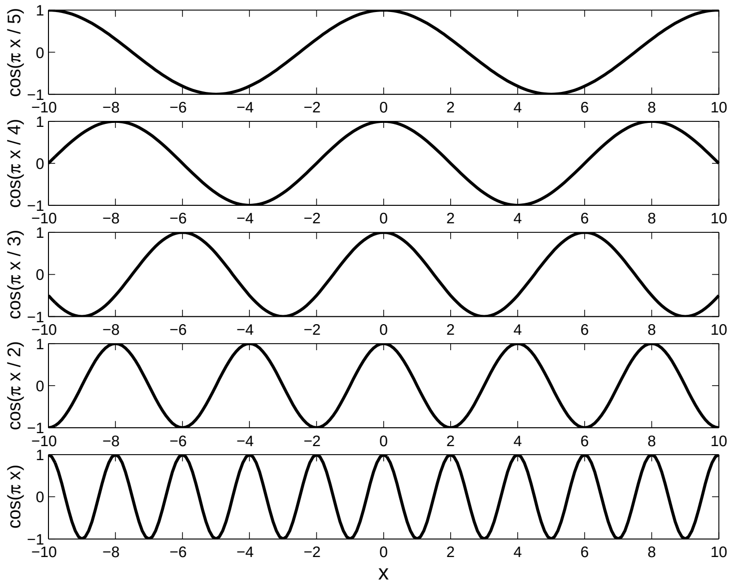


Figure 27.1:

Take cosines, of the form $cos(kx)$. Figure 27.1 shows different cosine waves. Each cosine wave has the form $\cos(kx)$, where $k = 2\pi/\lambda$ is the *wavenumber* and λ is the *wavelength.* At $x = \lambda$, $kx = 2\pi$, so $\cos(kx)$ is back to the maximum value, 1, that it had at $x = 0$. The wave $\cos(kx)$ has the same value at $x + \lambda$ as it had at x: $\cos[k(x + \lambda)] = \cos(kx + 2\pi) = cos(kx)$. Waves with different k and λ values (different wavelengths) have their maxima (other than the maximum at $x = 0$) and minima at different x. While they all add-up at $x = 0$, as we go away from $x = 0$, the different cosine waves *get out of phase.* This means that at a point x, some of the cosine functions for some k are positive, others are negative. The sum of cosines makes a *wave packet* concentrated at $x = 0$. Unless different k values are selected to satisfy very special relations, such that the different waves are *harmonics* of each other (see Problem 29.4), the sum of many cosines with a variety of k values will become zero as one goes away from the peak of the wave packet at $x = 0$.

The wave packet has a size Δx. *To make a sharp, small wave packet (Δx small), we need many cosines with different k.* The range of k's, Δk, must be large to make Δx small. As one expects intuitively, it can be shown that

$$\Delta k \Delta x \geq 1. \tag{27.1}$$

This is a fundamental property of all wave packets. The exact form of this relation depends on the definition of Δx, whether this quantity is defined as the value of x where the value of the total wave function is 1/2 the maximum value, or whether the square of the wave function is half-maximum, whether such criteria are applied at $x = \Delta x$ or at $x = \Delta x/2$ and so on. Similarly, there are different ways of defining the *bandwidth*, Δk. The important point is that no matter how the width Δx of the wave packet is defined, the width Δx is narrower in inverse proportion as the bandwidth Δk is wider. Δx and Δk are usually defined as the *standard deviations* of x and k using the relevant *intensity distribution* of the wave packet: the intensity is proportional to the absolute value squared of the wave function. We will not go into the details of this technical definition here, and these details are not important for this course. With these standard definitions, the width-bandwidth relation has the form,

$$\Delta k \Delta x \geq \frac{1}{2}. \tag{27.2}$$

Let us consider some examples to show that $\Delta k \Delta x \geq 1/2$ and Δx decreases for increasing Δk. Consider the following wave packets, each containing several waves of wavelengths close to $\lambda = 2\,m$:

$$\begin{aligned} f_3(x) &= \frac{1}{3}[\cos(\pi x) + \cos(1.02\pi x) + \cos(1.04\pi x)] \\ f_5(x) &= \frac{1}{5}\sum_{n=-2}^{2} \cos[(1 + 0.02n)\pi x] \\ f_9(x) &= \frac{1}{9}\sum_{n=-4}^{4} \cos[(1 + 0.02n)\pi x] \end{aligned} \tag{27.3}$$

Figure 27.2 shows these wave packets $f_3(x), f_5(x)$ and $f_9(x)$. $f_3(x)$ consists of 3 cosine waves of $\pi \leq k \leq 1.04\pi$, $f_5(x)$ consists of 5 waves of $0.96\pi \leq k \leq 1.04\pi$, and $f_9(x)$ consists of 9 waves of $0.92\pi \leq k \leq 1.08\pi$. As the bandwidth, Δk, increases, the width of the wave packet, Δx, decreases, as seen in Figure 27.2.

27.2 The Uncertainty Relation

The famous Heisenberg Uncertainty Relation of Quantum Mechanics is essentially a wave property. In the form, $\Delta k \Delta x \geq 1/2$, it is a well known relation for classical waves. This relation simply states the fact that the narrower a wave packet is (the smaller Δx), the larger the mixture of simple sine and cosine waves (the larger the variety of different wavelengths and wave numbers, the larger the bandwidth Δk) must be. In quantum mechanics, *wave-particle duality turns this statement into a statement about particle properties, momentum and*

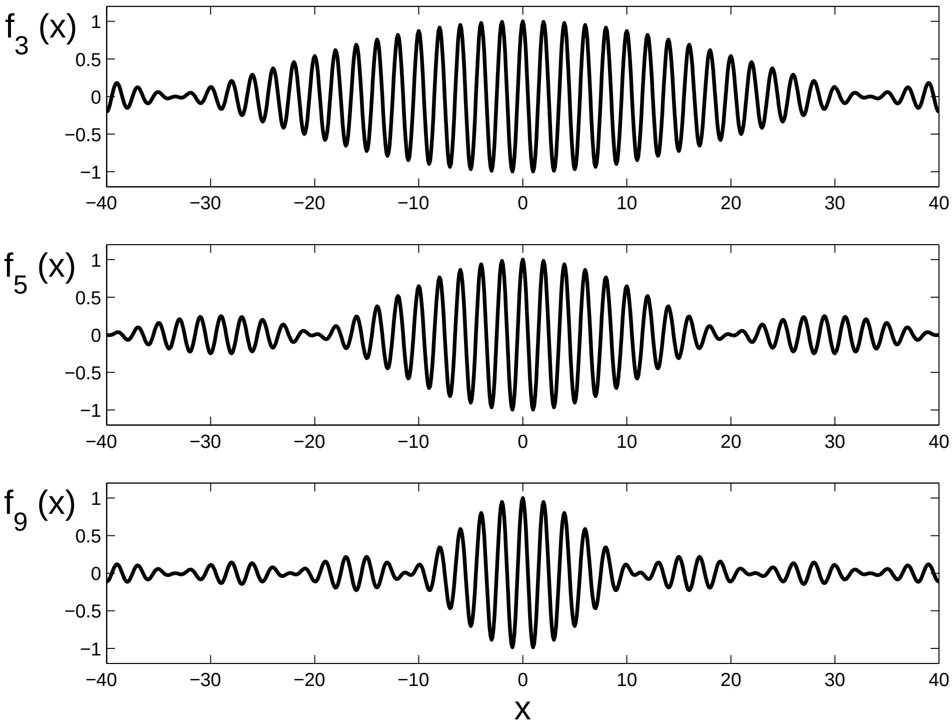


Figure 27.2:

position. Since de Broglie's relation states that $p = hk/(2\pi)$, the wave relation, Equation 27.2 becomes:

$$\boxed{\Delta p \Delta x \geq \frac{\hbar}{2} \qquad (27.4)}$$

This is ***The Uncertainty Relation***. It is very strange, and unlike the corresponding wave relation, Equation 27.2, it is totally counterintuitive as it refers to the momentum. But it is true according to all experiments! What makes this relation so strange in quantum mechanics is that it is applied to things like electrons, and to their particle properties like momentum p. Electrons are both waves and particles. This strange duality implies that $\Delta p \Delta x \geq \hbar/2$.

The Uncertainty Principle must be right: it comes about because electrons (and other classical "particles") are, at the same time, waves. The wave property, as we saw with the Bohr model, is absolutely essential for explaining why atoms have a certain size, why they have only certain energy values, and how they can be stable at all.

The electron is in a wave state that is, in general, not a pure cosine or sine wave. The wave function, describing the state of the electron (or any other non-relativistic quantum mechanical system) is obtained from a special wave equation, the *Schrödinger Equation*, just as sound waves, water waves, electromagnetic waves in vacuum, in glass, or in a plasma are solutions of their particular kinds of wave equations. The Bohr model contains the gist of wave-particle duality. It gives the right answers without containing all the details in any consistent manner, and without answering questions like where the electron is and how the atom can exchange energy with electromagnetic fields (*photons*). In fact, the Bohr model uses a semi-classical picture, being based on one particle condition (F = ma) and one wave condition (Bohr's Postulate). This picture is incomplete and inconsistent. The atom can be fully described only with the wave functions that are solutions of the Schrödinger wave equation.

What are these waves? The square of the wave function tells us the probability that the particle is here or there. The electron is spread out in space as a wave packet, a cloud. It is not moving on one of the Bohr orbits like a point particle. Instead it is a bit here and a bit there with some *probability* for being in each part of three dimensional space.

The meaning of the Bohr radius a_0 is simply the weighted average distance of the electron from the proton if the H atom is in the $n = 1$ energy state. Similarly, in the n^{th} state of the atom, the electron is at an *average* distance $r_n = n^2 a_0$ from the proton. The electron *is not at a definite place* at any moment in time! What is its momentum? Like all complicated wave packets, the electron wave is a superposition of many wavelengths, therefore *the electron has many momenta all at the same time.* As with position, it has a certain probability of this or that momentum value! The electron in any state has a certain range Δp of momentum values all at the same time!

With our daily experience, it does not feel strange to see a wave packet spread out in a region Δx of space, or to understand that a mixed wave, a wave packet, can be made up of simple waves of many different wavelengths (many "colours"). But when that wave is a particle, to have an electron spread out all over a certain region of the atom, of size Δx, and the fact that this electron has many momenta with a range Δp, all at the same time – this quantum picture of the electron is indeed very strange: it is not proper particle-like behavior at all!

27.3 Zero-Point Motion: Nothing Can Sit Still

A very important consequence of the Uncertainty Relation is that systems in stable equilibrium cannot be at rest, that the minimum energy state must include some kinetic energy. Consider a system near its stable equilibrium configuration. This could be, for example, a neutron moving around in the nucleus, or a chemical bond in a molecule. Let us consider a specific example:

> **Example 1:** Take the Cl_2 molecule. Say, the equilibrium length of the bond between the two Cl atoms is b. The potential energy of the total, complicated force between the two Cl atoms is approximately $U(x) = Kx^2/2$ where x is the difference between the interatomic bond length X and the equilibrium value b: $x = X - b$. If the system is in equilibrium, at $X = b$, $x = 0$, according to classical physics, it would be in the minimum energy state $U = 0$, and kinetic energy is zero if the atoms are not moving with respect to each other.
>
> *But the uncertainty principle says, there must be some amount of momentum, some amount of kinetic energy, kinetic energy cannot be zero!*
>
> To make the potential energy small, x must be small. But then the wave packet must be narrow, the spread of x values, Δx, must be small, $\Delta x \sim x$

to confine x to small values. This requires a mixture of many momenta. Even if $p = 0$ is one among the many momenta present, $p = 0$ cannot be the only momentum value: there must be a bandwidth, a range $\Delta p \cong \hbar/\Delta x \cong \hbar/x$ of momentum values (since we are only making an estimate, we do not keep the factor 1/2 in the Uncertainty Relation). The kinetic energy must be of the order of $E_K = \Delta p^2/2m \cong \hbar^2/(2mx^2)$. So as x becomes smaller, the potential energy becomes smaller but the kinetic energy increases! *According to the Uncertainty Principle, if something is pushed into a smaller region of space it has to move faster!*

So at what value of x is the total energy minimum? Using the Uncertainty Principle to eliminate the momentum dependence, the total energy can be expressed, approximately, as a function of x:

$$E(x) \cong \frac{\hbar^2}{2mx^2} + \frac{1}{2}Kx^2. \tag{27.5}$$

For estimating the minimum energy, one can use the minimum condition, $dE/dx = 0$, on the approximate energy $E(x)$. This gives

$$\frac{-\hbar^2}{mx^3} + Kx = 0$$

leading to

$$x = \left[\frac{\hbar^2}{Km}\right]^{1/4}$$

as the value of x, for which the total energy of the molecule will be minimum. In the minimum energy state the distance between the two Cl atoms is not the equilibrium distance b. Instead the distance values are spread around the value b in a range $x = [\hbar^2/(K\ m)]^{1/4}$! The corresponding minimum energy is found, by substituting this x value:

$$E_{\min} = \hbar\omega,$$

where $\omega = (k/m)^{1/2}$ is the oscillation frequency that the system near the minimum of the potential has (the exact value is $E_{\min} = \hbar\omega/2$. The Uncertainty Principle used in this simple way usually gives a fairly good estimate of the actual value of $E_{\min}$.)

We find that the Cl_2 *molecule (and any system near stable equilibrium) has a minimum energy that is more than the minimum of its potential energy. Nothing can "freeze" at the bottom of its potential energy curve. Everything must have some amount of kinetic energy!*

The Uncertainty Principle explains, in a similar way, why electrons do not just fall into the nucleus. According to classical physics electrons should just end up in the nucleus and there would be no atoms like the atoms we know, the atoms that we are made of!

Let us recall what classical physics predicts:
According to the laws of electromagnetism, when an electron moves around the nucleus on some orbit, the electric and magnetic fields it creates are changing all the time. The position of the charge and the location and direction of the current are changing as the electron moves around its orbit. Since the charge and current are the sources of the electric and magnetic fields, both fields must change with time. Maxwell's Equations imply that the changes in the fields are coupled to each other and will result in variations of the fields in space, such that electromagnetic waves will travel away from the atom in all directions. The electromagnetic waves must carry away energy, because when the waves arrive at a distant location where there are other charges, these charges will accelerate: this means that the electric field of the wave will do work on the distant charges. The wave fields have capacity to do work. So they must be carrying energy. This energy must come from the source of the electromagnetic wave, that is, the motion of the electron in the atom. In short, the atom must radiate electromagnetic waves, and therefore, it must lose energy.

What happens to the electron as it loses energy? As we saw, for $1/r^2$ forces like the electric force (and gravity), the particle kinetic energy is related to the potential energy. The potential energy is:

$$U(r) = -\frac{1}{4\pi\epsilon_0}\frac{e^2}{r}.$$

From the equation of motion for a circular orbit it is seen that $E_K = -U/2$. So the total energy is

$$E(r) = E_K + U = \frac{U}{2} = -\frac{1}{4\pi\epsilon_0}\frac{e^2}{2r}.$$

The total energy also decreases as r decreases: it becomes a larger negative number for smaller r. From one orbit, the electron will pass to a slightly smaller orbit. According to classical physics, all orbits at all radii and all energies are possible, so the energy will decrease continuously and the electron will follow a spiral path into the nucleus. In a very short time, the atom will disappear!

The wave picture for electrons is radically different. To fit in the electron wave, the atom can exist only in certain states, with discrete energy values E_n and definite shapes and sizes of the electron wave. The atom will still emit electromagnetic radiation. Maxwell's Equations are valid also in the quantum world. But in the process of emitting electromagnetic radiation the atom has to change from one of its possible states, say n, to another one, say n'. The atom can create electromagnetic waves carrying restricted packages of energy values, $\varepsilon = E_n - E_{n'}$. According to Planck's hypothesis, $\varepsilon = hf$, so radiation emitted by the atom will have only certain frequencies,

$$f = (E_n - E_{n'})/h. \tag{27.6}$$

The energy package or *quantum* of energy $\varepsilon = hf$ is called a *photon.* While electrons and other classical particles are at the same time waves, electromagnetic radiation fields, which are classically waves, are at the same time particles carrying these quanta of energy.

Electromagnetic radiation from other sources also effect the atom: the electron is charged so it will respond to electromagnetic fields. But this effect can only take place by changing the wave state of the electron: the atom can change from some state n to some other state

n' by absorbing a photon of energy $\varepsilon = E_n - E_{n'}$ and frequency $f = (E_n - E_{n'})/h$.

The lowest energy of the H atom is $E_1 = -13.6\ eV$. Why is there such a *lowest* energy? Why doesn't the H atom have a much lower energy state, where the radius of the atom (the average size of the electron wave) is much smaller, the electron wave packet is much closer to the proton?

After all this would have much lower electrostatic potential energy, due to the attraction between the proton and electron. If the electron did go into the nucleus, the potential energy of electrostatic attraction certainly would become much lower: the potential energy,

$$U(r) = -\frac{1}{4\pi\epsilon_0}\frac{e^2}{r},$$

is a larger negative number if the distance r between electron and proton is very small, say similar to the radius of the nucleus.

OK, we understand, in quantum mechanics, that the electron is a wave, and that the atom cannot just have any size and energy because it has to fit the wave. But the question is, why is the electron wave's lowest energy state still placing the electron so far from the proton? After all the Bohr radius $a_0 = 0.5\ Angstrom = 5 \times 10^{-11} m$ is some 100,000 times larger than the nuclear radius!

Let us do a small exercise, to use the Uncertainty Relation for finding the minimum energy that the H atom can have:

Example 2: To fit a wave into the atom, a certain sized region is needed. As we saw in Example 1, as the electron wave packet becomes smaller, the wave state must contain more wavelengths. So the wave state cannot have just $p = 0$, and there has to be a band of momentum values $\Delta p \cong \hbar/r$. The kinetic energy must be of the order of $E_K = \Delta p^2/2m \cong \hbar^2/(2mr^2)$. As r becomes smaller, the potential energy becomes smaller but the kinetic energy increases.

According to the Uncertainty Principle if something is pushed into a smaller region of space it has to move faster!

So at what value of r is the total energy of the H atom minimum?

$$E \cong \frac{\hbar^2}{2mr^2} - \frac{1}{4\pi\epsilon_0}\frac{e^2}{r}$$

For minimum energy, $dE/dr = 0$, so that

$$-\frac{\hbar^2}{mr^3} + \frac{1}{4\pi\epsilon_0}\frac{e^2}{r^2} = 0.$$

Solving this for r gives

$$r = 4\pi\epsilon_0\frac{\hbar^2}{me^2} = a_0$$

and so

$$E_{\min} = -\frac{1}{4\pi\epsilon_0}\frac{e^2}{2a_0} = -13.6\ eV$$

is obtained.

The tradeoff between kinetic energy increasing and potential energy decreasing as r decreases sets the minimum size and energy of the H atom to these values, so the electron wave does not get any closer to the nucleus. The use of the Uncertainty Relation usually leads to estimates close to the exact value. In this case, the estimate happens to give the exact value obtained from the Bohr model and from the Schrödinger Equation for the electron wave.

CHAPTER 27 - PROBLEMS:

1. (a) What are the typical values of distances and momenta in daily life? What are typical uncertainties in distance and momentum measurements in daily life?

   (b) Why are we not aware of the Uncertainty Relation $\Delta p \Delta x > \hbar/2$ in our daily life?

2. An electron in a molecule has potential energy $U(x) = Cx^3$ where C is a constant, $C = 8$ *Rydberg Angstrom*$^{-3}$. Estimate the minimum energy of this electron.

3. **Quantum Mechanical Levitation:** A particle of mass m is *in vacuum* in the gravitational potential of the Earth, given approximately as $U(z) = mgz$ at distance z above a smooth solid surface at $z = 0$, for z much less than the radius of the Earth; here $g \cong 10\ m\ s^{-2}$ is the gravitational acceleration near the Earth's surface.

   (a) Estimate the distance z of the particle above the Earth's surface, and its minimum energy, in terms of m, g and $\hbar$.

   (b) Evaluate the levitation height in vacuum on the surface of the Earth for an electron, a proton, and for yourself.

   (c) What happens in real situations? Answer: In any situation but absolute vacuum, electromagnetic fields from other particles, and resulting contributions to the potential energy, are much much larger than the gravitational potential energy due to the Earth, which is totally negligible. But in any realistic situation, all particles *are "levitating", and are in motion*, in the atomic potentials, since according to the Uncertainty Principle they cannot "sit still" in the bottom point of the potential.

4. **Harmonics:** Two cosine waves with different wavelengths λ_1 and λ_2 both have maxima at $x = L$. What is the relation between λ_1 and λ_2?

   A wave packet that contains only wavelengths that are harmonics of each other will repeat at L, 2L, 3L etc. A wave packet that is concentrated in one region, with a single peak, cannot be made of only wavelengths that are harmonics of each other.

5. (*) The superposition of *only two* cosine (or sine) waves does not produce a localized wave packet. *Beats* are produced:

   (a) Consider the superposition of two cosine waves,

$$f_1(x) = \frac{1}{2}[\cos(k_1 x) + \cos(k_2 x)]$$

Using the trigonometric identities

$$\begin{aligned}\cos(A+B) &= \cos A\cos B - \sin A\sin B\\ \cos(A-B) &= \cos A\cos B + \sin A\sin B\end{aligned}$$

show that

$$f_1(x) = \cos[(\frac{k_1-k_2}{2})x] \times \cos[(\frac{k_1+k_2}{2})x]$$

(b) Using this formula, plot

$$f_1(x) = \frac{1}{2}[\cos(\pi x) + \cos(1.02\pi x)]$$

for $-200 \leq x \leq 200$. For oscillations as a function of time, the superposition of sound waves at two frequencies ω_1 and ω_2 that are very close to each other is a note at the average frequency $1/2(\omega_1+\omega_2)$ modulated at the very low frequency $1/2(\omega_1-\omega_2)$. The intensity of the sound wave, which is proportional to the *square* of the oscillations in air pressure, will contain the *beat frequency* $(\omega_1-\omega_2)$ which you hear as *beats* when a pair of strings on a musical instrument are slightly out of tune.

Chapter 28

The Pauli Exclusion Principle

Suppose there is a single electron in a wave state $\Phi(x_1)$ in some quantum mechanical system, as seen in Figure 28.1.

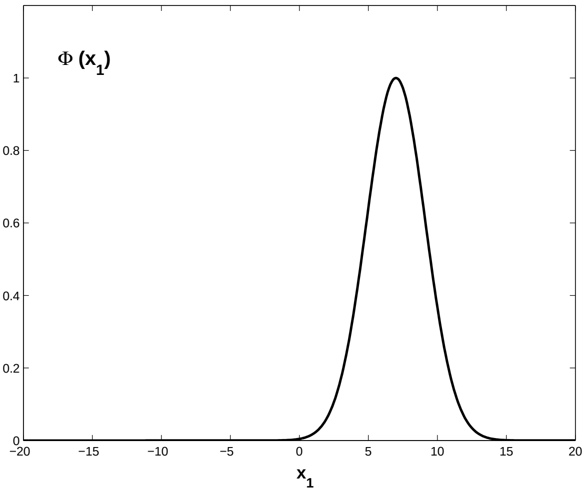


Figure 28.1:

The probability of finding this electron at position x_1 within a small neighborhood of dx_1 is:

$$P(x_1)dx_1 = |\Phi(x_1)|^2 dx_1.$$

If there is a single electron with a different wave state $\Psi(x_2)$ as seen in Figure 28.2, the probability of finding this electron at position x_1 within a small neighborhood of dx_1 is:

$$P(x_2)dx_2 = |\Psi(x_2)|^2 dx_2.$$

If we have two electrons, one in the wave state $\Phi(x_1)$ and another in the wave state $\Psi(x_2)$, the wave state of both electrons must be something like:

$$\Phi(x_1)\Psi(x_2),$$

as seen in Figure 28.3, such that the probability of finding one electron at x_1 and the other

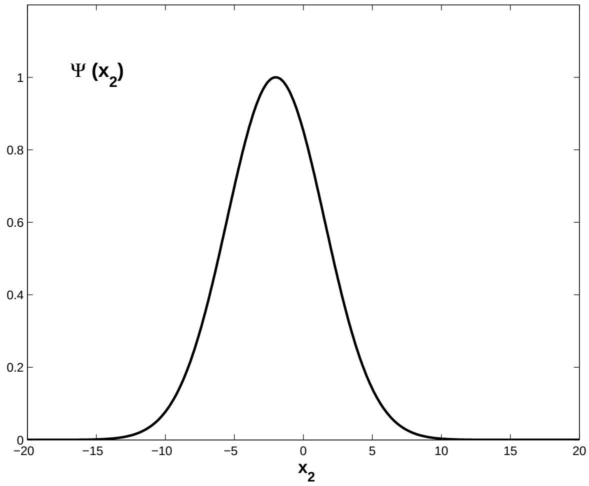


Figure 28.2:

at x_2 is $|\Phi(x_1)\Psi(x_2)|^2 = |\Phi(x_1)|^2|\Psi(x_2)|^2$, as seen in Figure 28.4.

The probability of two different things happening independently is the product of their individual probabilities. Just like, when you throw two *dice*, the probability of getting 4 on one of them is 1/6, the probability of getting 3 on the other die is also 1/6. The probability of getting 4 on the first die and 3 on the second die is

$$\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}.$$

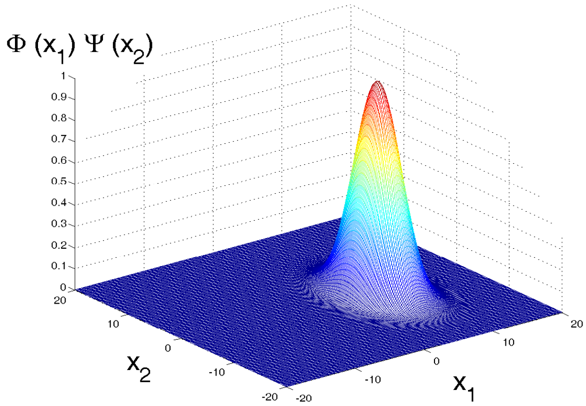


Figure 28.3:

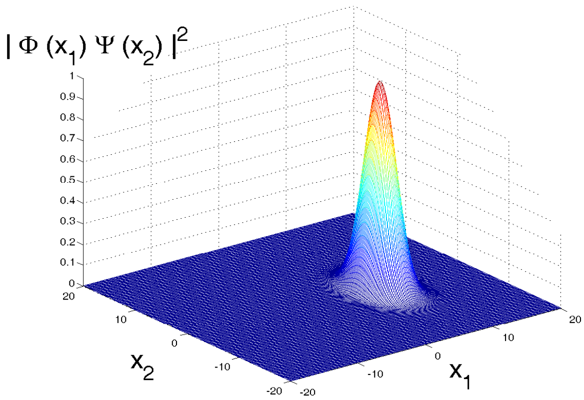


Figure 28.4:

But all we know is that <u>one</u> electron is in the wave state Φ and <u>another one</u> in the state Ψ. So why not use a wave function $\Phi(x_2)\Psi(x_1)$ (Figure 28.5)?

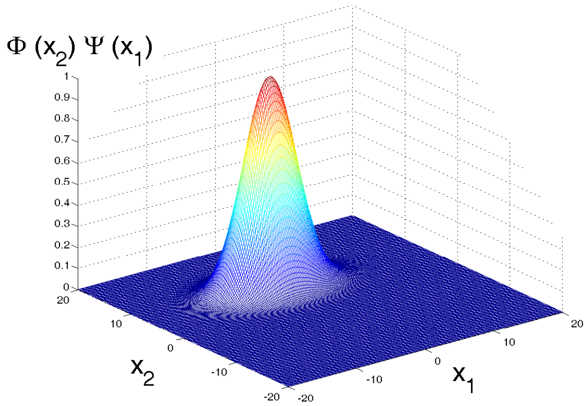


Figure 28.5:

The associated probability is $|\Phi(x_2)\Psi(x_1)|^2 = |\Phi(x_2)|^2|\Psi(x_1)|^2$ (Figure 28.6):

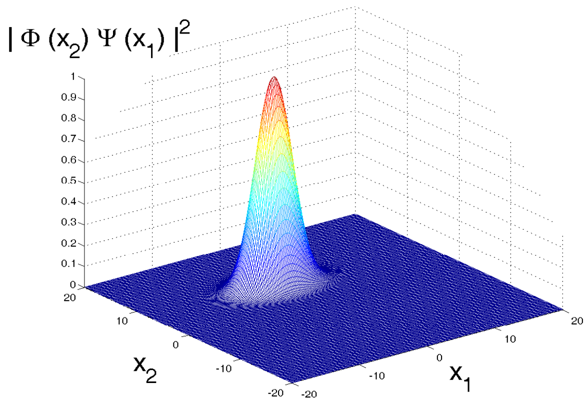


Figure 28.6:

We cannot distinguish the two electrons. They are identical. In classical physics we can distinguish them – we label them as the one at x_1 and the other at x_2. But when they are in wave states, we cannot label them with x_1 and x_2, because *waves do not have exact positions.* The variables x_1 and x_2 are not specific positions to label and distinguish the electrons.

Neither $\Phi(x_1)\Psi(x_2)$ or $\Phi(x_2)\Psi(x_1)$ can be the correct wave function for two electrons.

The correct wave function can be perhaps:

$$f_+(x_1, x_2) = \Phi(x_1)\Psi(x_2) + \Psi(x_1)\Phi(x_2)$$

as seen in Figure 28.7:

The probability to have this state is $|f_+(x_1, x_2)|^2$ and it is shown in Figure 28.8.

OR,

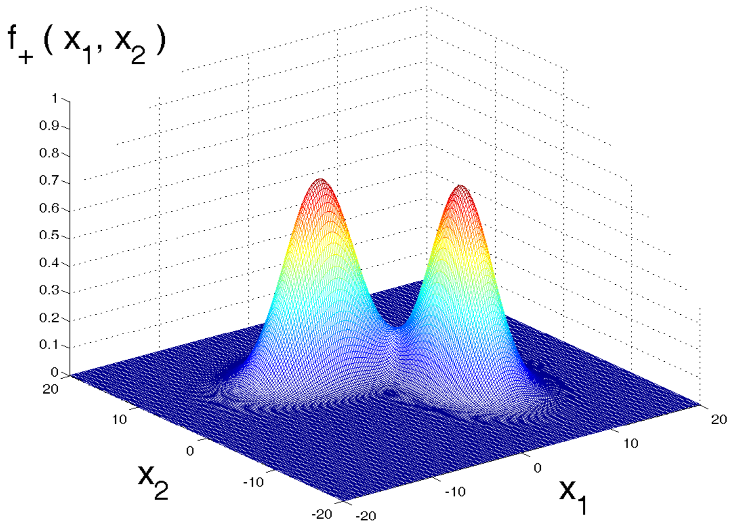


Figure 28.7:

There is another mathematical possibility:

$$f_-(x_1, x_2) = \Phi(x_1)\Psi(x_2) - \Psi(x_1)\Phi(x_2)$$

which is seen in Figure 28.9.

The sign '–' is OK, because *the probability* $|f_-(x_1, x_2)|^2$ *does not distinguish* x_1 *and* x_2, as seen in Figure 28.10.

> So, which of the two mathematical possibilities for wave states of <u>two</u> identical electrons is the one that actually occurs in nature?

To decide this we need a prediction to test either hypothesis, the f_- or f_+ rules for combining the wavestates of two electrons.

Let us assume that f_- is the correct rule for electrons.

This implies that *we cannot place two electrons in the same wave state*: If Φ and Ψ are the same functions the result is:

$$f_-(x_1, x_2) = \Phi(x_1)\Psi(x_2) - \Psi(x_1)\Phi(x_2)$$

$$f_-(x_1, x_2) = \Phi(x_1)\Phi(x_2) - \Phi(x_1)\Phi(x_2)$$

$$f_-(x_1, x_2) = 0$$

> Therefore, the prediction of using the f_- wave combination is: *Two electrons are never in the same state.* This is called the "***The Pauli Exclusion Principle***".

Chemistry and Atomic Physics show, experimentally, that indeed in any atom there are no two electrons in exactly the same state. The structure of atoms and the entire periodic table is the experimental proof that the Pauli Exclusion Principle holds for electrons. So, the correct choice of wave state for two electrons is:

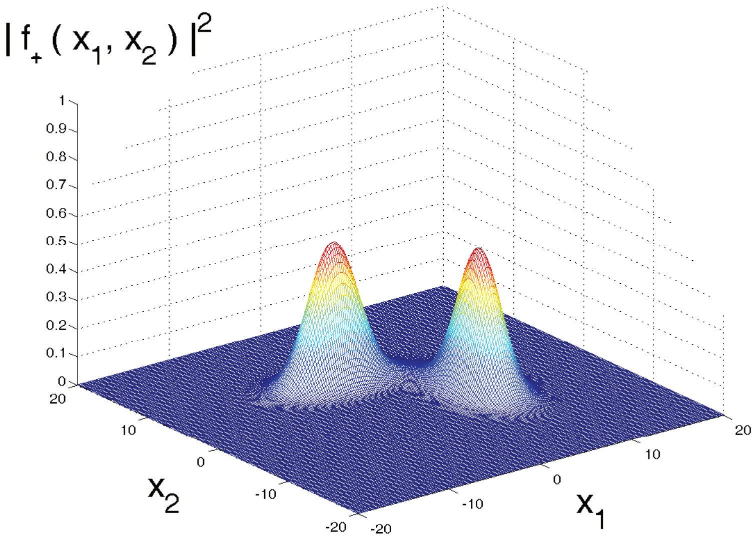


Figure 28.8:

$$f_-(x_1,x_2) = \Phi(x_1)\Psi(x_2) - \Psi(x_1)\Phi(x_2).$$

The Pauli Principle applies for protons and neutrons too. In nuclei we find that no two protons are in exactly the same wave state, and no two neutrons are in exactly the same wave state.

Do all kinds of particles obey the Pauli Principle, the f_- rule of combining single particle wave functions into two particle wave functions? It turns out that in Nature, there are also many kinds of particles that obey the f_+ rule. The most important example is the photon. Other examples are some kinds of sub-nuclear fundamental particles called mesons.

A kind of particle that obeys the f_+ rule has just opposite property to that stated with the Pauli Principle. For f_- particles, the Pauli Principle states that if there is already one particle in a given state a second particle cannot get into that same state. The f_+ rule implies just the opposite: if there is already one f_+-type particle in a given state, a second particle coming into the system is more likely to be in the same state. If there are many photons already in the same state (same frequency, same wavelength, traveling in the same direction, same *polarization*) then it is most likely that atoms will emit more photons into that same state. This is the principle of the *laser*[1]. Other results of the f_+ rule are *Bose-Einstein condensation, superfluidity and superconductivity.*

Particles that obey the f_- rule are called ***fermions***, in honour of Enrico Fermi.

Particles that obey the f_+ rule are called ***bosons***, in honour of Satyendranath Bose.

What decides whether a given type of particle is a boson or fermion is its angular momentum. Experiments show that angular momentum is quantized, like energy: any particle can have angular momentum that is either an integer multiple of $\hbar$, like $0, \hbar, 2\hbar, 3\hbar, \ldots$ or a half-integer value, like $\hbar/2, 3\hbar/2, 5\hbar/2, \ldots$. Single fundamental particles have an intrinsic angular momentum property called *spin.* The electron, proton and neutron each have spin values of $\hbar/2$. They are "*spin 1/2*" particles in units of $\hbar$. Photons have spin $\hbar$ – they are

[1]The word "laser" is just short for "Light Amplification by the Stimulated Emission of Radiation".

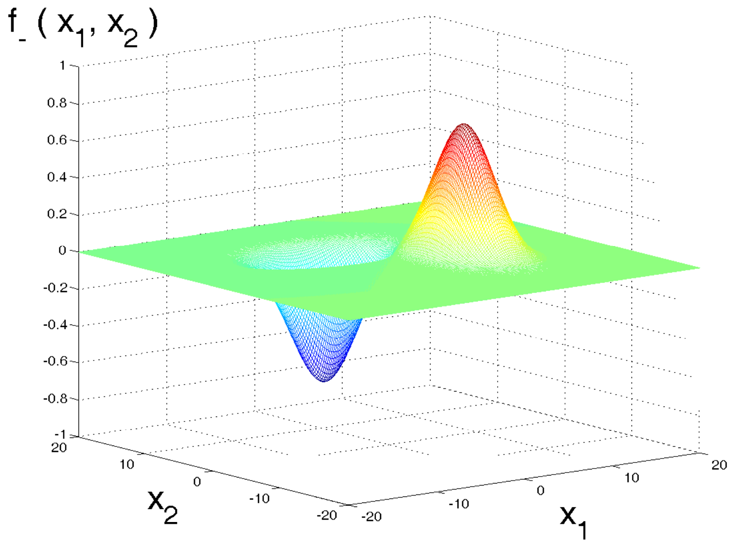


Figure 28.9:

"*spin one*" objects. For composite systems, like atoms, the total angular momentum includes contributions from orbital angular momenta and spins of the constituent particles.

> Particles or systems of particles with half-integer values of angular momentum are fermions. They obey the f_- rule, the Pauli Exclusion Principle.
> Particles or systems of particles with integer values of angular momentum are bosons. They obey the f_+ rule, and tend to all get into the same state.

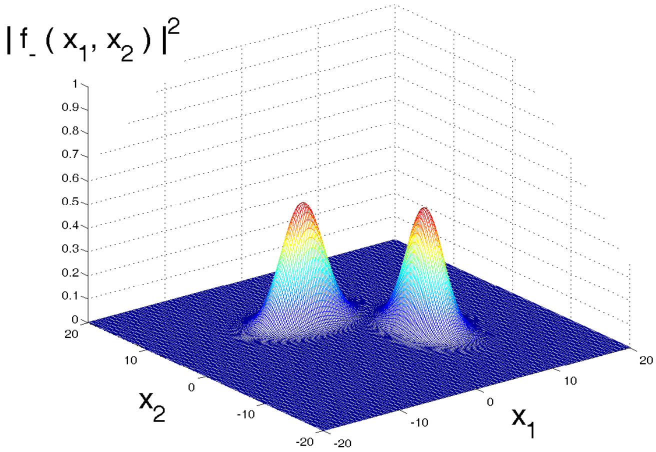


Figure 28.10:

CHAPTER 28 - PROBLEMS:

1. The possible wave functions for the particle-in-a-box (quantum box) problem are $\psi_n(x) = (2/L)^{1/2}\sin(n\pi x/L)$ where L is the size of the box and n is an integer.

   (a) There are two neutrons in the box, one in the state n = 3, and one in the state n = 4. What is the two-particle wave function $f(x_1, x_2)$ for the two neutrons?

   (b) There are two photons in the box, one in the state n = 1, and one in the state n = 5. What is the two-particle wave function $f(x_1, x_2)$ for the two photons?

2. The energy of a particle in the n^{th} wave state for the particle-in-a-box (quantum box) problem is $E_n = \hbar^2 n^2 \pi^2 \ / \ (2m\ L^2)$ where L is the size of the box.

   (a) What is the lowest possible total energy for nine neutrons in the box?

   (b) What is the lowest possible total energy for nine photons in the box?

3. (*) Three identical fermions are in the wave states described by the wave functions ψ_1, ψ_2 and ψ_3. What is the wave function $f_-(x_1, x_2, x_3)$ for the three fermions?

4. (*) Three identical bosons are in the wave states described by the wave functions ψ_1, ψ_2 and ψ_3. What is the wave function $f_+(x_1, x_2, x_3)$ for the three bosons?

Chapter 29

Wave Shapes for Atomic Electrons

As we saw with the Bohr model, only certain discrete values of energy are possible for an electron to exist as a wave packet in an atom. These energy values E_n are labeled with an integer $n = 1, 2, 3, ..$, called the principle quantum number. $n = 1$ is the wave state with the lowest energy.

It turns out that in real atoms in three dimensional space there are different possible shapes and orientations of the electron wave packet that have the same energy E_n. The differences are related with the amount of rotation (angular momentum) that the electron wave packet (cloud) has. This is labeled l, and called the angular momentum quantum number. For the principle quantum number n, there are several possible values of l corresponding to electron wave packets (clouds, orbitals) of different shapes: $l = 0, 1, 2, ...,\ n-1$ are the different possibilities. $l = 0$ wave packets are spherically symmetric with respect to the nucleus – for zero angular momentum, zero rotation, the shape is not distorted. The higher the l value, the more distorted the wave packet is.

For each value of l, there are several distinct possibilities for shapes and orientations of the wave packet in three dimensional space. These orientations are labeled by a quantum number m. For each l, there are $2l + 1$ different possible wave packet orientations $m = l, l-1, ..., 2, 1, 0, -1, -2, ..., -l$.

For $l = 0, m = 0$: for a spherical shape there is only one way to orient the wave. $l = 0$ states are also denoted as s orbitals.

For $l = 1$, there are 3 possibilities: $m = 1, 0$ or -1, because $l = 1$ wave packets are cylindrically symmetric shapes around some axis, and there are three distinct ways to orient these wave packets in space, for example, along the x or y or z axes. $l = 1$ wave packets are also called p orbitals.

For $l = 2$, there are 5 possibilities: $m = 2, 1, 0, -1$ or -2. $l = 2$ wave packets are also called d orbitals.

The reason for only discrete wave shapes l, m to be possible is the same as the reason as why only certain energies E_n are possible: electrons are waves, and to fit waves into a system like the attractive potential energy well of the atom requires certain conditions. Only special wavelengths and wave shapes will fit a given system. We saw this in our Experiment 3 for waves on a string. We obtained the selected E_n and r_n values for the Hydrogen atom correctly from the Bohr model, applying Bohr's condition of fitting wavelengths into an orbit. For other atoms, and for obtaining the l, m values and the correct waveshapes, one must solve the Schrödinger Wave Equation. We will not do this for waveshapes in atoms in this course, but we understand the physical reason that there can be only so many certain waveshapes as a natural consequence of the wave properties of electrons.

There is another property of the electron, called the *spin*, which is like an intrinsic angular momentum or like a tiny magnet that the electron carries. Experiments show that an electron spin can have only two values, $+1/2$ or $-1/2$ in units of $\hbar$. Sometimes these two distinct possibilities are labeled as "up" and "down", and denoted with up or down pointing little arrows.

How many different wave packet and spin states are there with the same energy, for a given n? Add up the number of different m's for each l, that is $2l+1$ different m values, for all the different values of l from $l = 0$ to $l = n-1$. This sum has the value n^2. Then in each possible wave state, the electron has two different possibilities for spin. So altogether there are $2n^2$ different possible wave packet and spin states for electrons of energy E_n.

Quantum Number	**Values**	**Range**	**Notes**
n	1,2,3,...	$n \geq 1$	
l	0,1,2,...,$n-1$	$0 \leq l \leq n-1$	$l = 0:$ s orbital $l = 1:$ p orbital $l = 2:$ d orbital
m	$-l, ..., -1, 0, 1, ..., l-1, l$	$-l \leq m \leq l$	

Table 29.1: Summary of Quantum Numbers

Now consider an atom with Z electrons. If there was no Pauli Principle, the lowest total energy state of the atom would be for all Z electrons to have the lowest energy E_1 possible for a single electron. But the Pauli Principle does not allow more than one electron in the same wave packet and spin state, so it allows only 2 electrons in the $n = 1$ state with energy E_1. To make the total energy as low as possible, 2 electrons are in the $n = 1$ state; then as many as possible, namely $2n^2 = 8$ electrons in the next lowest energy state E_2, $n = 2$, and so on until all Z electrons are placed in the possible energy states with the lowest energies. For a given n, the lower l values actually have slightly lower energies, so the filling starts with $l = 0$ states, called *s states* and goes on to $l = 1$, called *p states*, and then to $l = 2$, called *d states*, etc.

For $Z = 7$, Nitrogen, the resulting distribution is $1s^2\ 2s^2\ 2p^3$. The meaning of this notation: the $n = 1, l = 0$ state (s) has 2 electrons; the $n = 2, l = 0$ state (s) has 2 electrons; and the $n = 2, l = 1$ state (p) takes up the remaining 3 electrons.

<table>
<tr><th>n</th><th>1</th><th colspan="4">2</th><th colspan="9">3</th></tr>
<tr><td>l</td><td>0</td><td>0</td><td colspan="3">1</td><td>0</td><td colspan="3">1</td><td colspan="5">2</td></tr>
<tr><td>m</td><td>0</td><td>0</td><td>-1</td><td>0</td><td>1</td><td>0</td><td>-1</td><td>0</td><td>1</td><td>-2</td><td>-1</td><td>0</td><td>1</td><td>2</td></tr>
<tr><td>N_{e^-} ($= 2n^2$)</td><td>2</td><td colspan="4">8</td><td colspan="9">18</td></tr>
<tr><td>e^-</td><td>$1s^2$</td><td colspan="4">$2s^2\ 2p^6$</td><td colspan="9">$3s^2\ 3p^6\ 3d^{10}$</td></tr>
</table>

Table 29.2: Possible n and l values, total number of electrons, and electron distributions for $n = 1, 2, 3$

In atoms with many interacting electrons, the order of energies of n, l states may change.

For example, $4s$ energy levels have lower energy than $3d$. Taking all this into account, and filling the available energy levels up starting from the lowest energy wave state according to the Pauli Principle gives an understanding of the structure of the **Periodic Table**.

The chemical properties of each element are determined by the number of electrons in its highest energy shell (highest n), because these are the electrons that are easiest to remove from the atom, by giving the smallest amount of energy. Similarly, the number of empty places in the last shell (how many more electrons can the atom take from other atoms) plays a role in its chemical properties. The ways of filling up the energy levels, according to the Pauli Principle, determine the properties for each atom with Z electrons. Therefore, the Periodic Table itself is the experimental proof of the Pauli Principle.

Finally, note that the energy values, and the number of different possible wave packets for each energy value (the n, l, m assignments), are different for each system containing electrons (or protons, or neutrons, or other particles obeying the Pauli Principle). But the application of the Pauli Principle is the same: First, list all possible energy values for $n = 1, 2, 3, ...$ for a single electron in that system. These values and their dependence on n is different in each system. Then, find how many different wave states $w(n)$ exist for each n. Then, place $2w(n)$ electrons in each state starting from $n = 1$, until the total number N of electrons in the system are all placed. (For atoms, $w(n) = n^2$).

CHAPTER 29 - PROBLEMS:

1. What is the distribution of the 9 electrons among the different n, l, m and spin s states for the lowest total energy state (ground state) of the Fluorine atom?
2. What is the distribution of the 10 electrons among the different n, l, m and spin s states for the ground state of the Neon atom?
3. What is the distribution of the 11 electrons among the different n, l, m and spin s states for the ground state of the Sodium atom?
4. What is the distribution of the 26 electrons among the different n, l, m and spin s states for the ground state of the Iron atom?

Exam Problems - Quantum Mechanics

1. **[Spring 2007, Final]** A particle of mass m is moving in a harmonic oscillator potential $U(x) = \frac{1}{2}kx^2$.

   (a) Use the Uncertainty Principle to express the kinetic energy in terms of x.

   (b) At what value of x is the total energy minimum?

   (c) What is the minimum value of the energy in terms of k and m?

   (d) Express your estimate of minimum energy in (c) in terms of the angular frequency ω of the harmonic oscillator.

2. **[Fall 2006, Final]** An electron in a molecule has a potential energy $U(x) = Cx^4$ where x denotes the distance from the equilibrium position. Use the Uncertainty Principle $\Delta p \Delta x \sim \hbar/2$ to estimate

   (a) the total energy of the electron as a function of x.

   (b) the value of x, where the total energy of the electron is minimum.

   (c) the minimum value of the total energy.

   (d) the amount of kinetic energy when the total energy is minimum.

   (e) the amount of potential energy when the total energy is minimum.

3. **[Fall 2006, Final]** In one dimension, two particles of identical mass $m_1 = m_2 = m$ in gravitational field g interact with a harmonic potential so that the total energy of the system is given by

$$E = \frac{p_1^2}{2m} + mgx_1 + \frac{p_2^2}{2m} + mgx_2 + \frac{1}{4}m\omega^2(x_1 - x_2)^2.$$

   The particles are confined to move above a rigid surface so that $x_1 > 0$ and $x_2 > 0$ (see Figure 29.1).

   (a) Show that the total mechanical energy can be rewritten as

$$E = \frac{P^2}{2M} + MgX + \frac{p^2}{2\mu} + \frac{1}{2}\mu\omega^2 x^2$$

   where $M = 2m$ is the total mass and $\mu = m/2$ is called the reduced mass. Here, $X = (x_1 + x_2)/2$ is the position of the center of mass and $x = x_1 - x_2$ is the relative coordinate. The corresponding momenta are $P = M\frac{dX}{dt} = p_1 + p_2$ and $p = \mu\frac{dx}{dt} = p_1 - p_2$, respectively.

(b) Assume that the center of mass coordinate X and the relative coordinate x can be treated independently. Using the uncertainty principle

$$\Delta p \Delta x \sim \hbar/2,$$

find the lowest possible values for $E_X = \frac{P^2}{2M} + MgX$ and $E_x = \frac{1}{2}\mu\omega^2 x^2$ to estimate the ground state energy $E = E_X + E_x$.

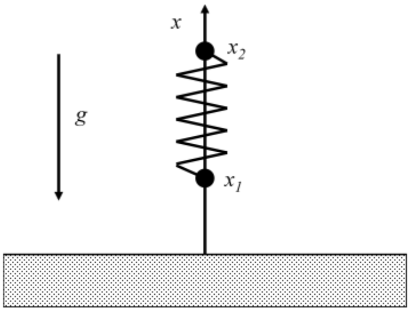


Figure 29.1:

4. **[Fall 2006, Final]** A particle of mass m is trapped in one-dimensional box of length L, as shown in Figure 29.2. The potential energy $U(x) = 0$ inside the box, at $0 < x < L$ and is very large ("infinite") outside the box, so that the particle cannot be outside the box at all.

   (a) Thinking of the wave function of the particle as a sine wave, with fixed ends, find the wavelengths λ_n that are allowed in this problem.

   (b) Use the de Broglie relation $\lambda_n = h/p_n$ to find the allowed momenta.

   (c) Use $E_n = p_n^2/2m$ to find the allowed energies.

   (d) What is the frequency of a photon that is emitted as the particle drops from level $n = 4$ to level $n = 2$?

   (e) Using the Pauli exclusion principle obtain the total energy of three electrons placed in this potential.

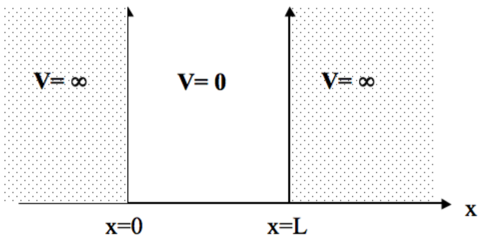


Figure 29.2:

5. **[Fall 2009, Final]** The Li^{++} ion consists of a nucleus of charge $+3e$ and one electron.

   (a) Write the equation of motion $F = ma$ assuming the electron is in a circular orbit of radius r, and using the electrostatic force

   $$F = \frac{1}{4\pi\epsilon_0}\frac{3e^2}{r^2}.$$

   (b) Write Bohr's postulate for the relation between the electron wavelength λ and the radius r of the orbit. Express this as a relation between electron velocity and radius of the orbit, using the de Broglie relation $\lambda = h/mv$.

   (c) Solve the two equations from (a) and (b) to find the values of r.

   (d) What are the possible energy values of the Li^{++} ion?

6. **[Fall 2009, Final]** An electron is trapped in a two-dimensional square box with sides of length L. The potential energy $U(x) = 0$ inside the box, for $0 < x < L$ and $0 < y < L$. The potential energy is very large ("infinite") outside the box, so that the electron cannot be outside the box at all (see Figure 29.3).

   (a) The wave functions of the particle are sine waves, $\psi(x, y) = A\sin(k_x x)\sin(k_y y)$. Using $\psi = 0$ if $x = L$ or if $y = L$, find the wave numbers k_x, k_y that are allowed in this problem.

   (b) The wavevector for the two-dimensional box is given by $\mathbf{k} = k_x\mathbf{i} + k_y\mathbf{j}$. Find the allowed energies $E = \hbar^2k^2/2m$ of the electron, in terms of k_x and k_y.

   (c) For each allowed (k_x , k_y) wave there are two spin states. How many different electron states correspond to each of the lowest three energy levels?

   (d) 7 electrons are placed in this square box. Use the Pauli exclusion principle to find the lowest possible total energy of the system.

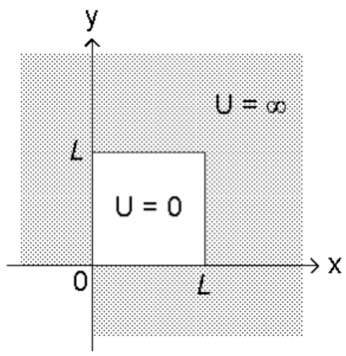


Figure 29.3:

7. **[Fall 2008, Final]** A particle of mass m is moving in a potential energy $U(x) = Ax^6$ where A is some constant.

   (a) Express the total energy in terms of x only, by using the Uncertainty Principle $\Delta p \Delta x \approx \hbar$.

(b) Find the value of x for which the total energy is minimum.

(c) Find the minimum value of the energy.

8. **[Fall 2008, Final]** A particle of mass m is moving in a circular orbit of radius r. The potential energy of the particle $U(r) = Cr^3$, where C is some constant.

   (a) Find the force corresponding to this potential energy.

   (b) Write the two equations of the Bohr model for this potential energy.

   (c) Find the allowed values of the radius r of the orbit.

   (d) Find the allowed values of the total energy.

9. **[Fall 2007, Final]** A muon is an elementary particle with the same charge as the electron, $-e$, and a mass that is 200 times the electron's mass, $m = 200m_e$. A muon can be captured by a proton to form a "muonic atom". Using the Bohr postulate $2\pi r = n\lambda = nh/mv$, where $n = 1, 2, 3...$,

   (a) Find the Bohr radii r_n for the muonic atom in terms of mass m, charge e and fundamental constants.

   (b) Find the possible energy levels in terms of m, e and fundamental constants.

   (c) Find the energy and the frequency of the photon emitted when the muonic atom makes a transition from $n = 3$ to $n = 2$ energy state in terms of the ground state energy E_1.

10. **[Fall 2007, Final]** The potential energy of a neutron in a nucleus as a function of its distance r from the center of the nucleus is approximately $U(r) = \frac{1}{2}Kr^2$.

    (a) Use the Uncertainty Principle to estimate the radius for which the energy is minimum.

    (b) What is the minimum energy E_{min} of the neutron?

    (c) The first excited state energy of a <u>one dimensional</u> harmonic oscillator is 3 times the lowest energy, $E_2 = 3E_1$. Noting that $r^2 = x^2 + y^2 + z^2$, what are the lowest three possible energies of the neutron in terms of the minimum energy E_{min}?

    (d) What is the degeneracy of each of these 3 lowest energy levels? (Degeneracy means how many physically different states have the same energy)

    (e) Neutrons have two spin states and obey the Pauli Principle. What is the minimum total energy of 8 neutrons placed in this nucleus, in terms of E_{min}.

11. **[Fall 2003, Final]** A neutron of mass m is bound in a nucleus. The potential energy is $U(r) = \frac{1}{2}kr^2$ in terms of the distance r from the center of the nucleus. The neutron has zero angular momentum, therefore its motion is entirely in the radial direction. Use the Uncertainty Principle, $p_r \geq \hbar/r$, to express the neutron's total energy in terms of r only, and estimate the minimum energy that the neutron must have.

12. **[Fall 2003, Final]** The Bohr model can be used to find the energy levels for any potential $U(r)$. Consider an electron of mass m bound by a potential of the form $U(r) = \frac{1}{2}kr^2$, where r is the distance from the origin.

(a) Find the force and write the equation of motion for circular orbits for the electron.

(b) Find the possible orbital radii r_n according to the Bohr model.

(c) Find the possible energy levels for this electron according to the Bohr model.

(d) The electron makes a transition from the lowest energy level to the next energy level by absorbing a photon. What is the energy of the photon?

(e) What is the photon's frequency?

13. **[Fall 2003, Final]** N molecules of an ideal gas are in thermal equilibrium at temperature T inside a container of volume V.

(a) Find the average kinetic energy.

(b) If the mass of each molecule is m, what is the average momentum $\langle p \rangle$?

(c) What is the average de Broglie wavelength λ_{dB}?

(d) Assuming each particle is really a wave packet of approximate volume λ_{dB}^3, find the temperature T_c at which the wave packets begin to touch and the classical ideal gas model becomes inapplicable.

14. **[Spring 2010, Final]** According to the Bohr's model of hydrogen atom, the radius of possible electron orbit is given by

$$r_n = n^2 a_0$$

where a_0 is the Bohr radius and $n = 1, 2, 3, \ldots$

(a) Using Bohr's postulate and de Broglie relation, find the orbital speed of the electron, v_n, in terms of a_0.

(b) What is the kinetic energy of the electron in terms of a_0 in these possible orbits?

(c) What is the total energy of the electron in terms of a_0 in possible orbits?

(d) What is the frequency of the photon that would be emitted if the electron transits from $n = 3$ state to $n = 1$ state?

15. **[Spring 2010, Final]** Consider a hydrogen atom: a proton in the nucleus and an electron at the lowest energy orbit.

(a) Write down the total energy of the electron.

(b) Using uncertainty principle, express the total energy as a function of r (that is the distance between the two charges).

(c) Sketch the total energy as a function of r.

(d) Calculate the minimum energy $E_{\min}$ of the electron.

(e) What is the physical significance of $r_{\min}$ (that is the distance between the two charges at which the total energy of the electron is its minimum)?

16. **[Fall 2010, Final]** The He^+ ion consists of a nucleus of charge $+2e$ and one electron.

(a) Write the equation of motion $\mathbf{F} = m\mathbf{a}$, assuming the electron is in a circular orbit of radius r, and using the electrostatic force,

$$F = -\frac{1}{4\pi\epsilon_0}\frac{2e^2}{r^2}\hat{\mathbf{e}}_{\mathbf{r}}$$

(b) Write Bohr's postulate for the relation between the electron wavelength λ and the radius r of the orbit. Express this as a relation between electron velocity and radius of the orbit, using the de Broglie relation $\lambda = h/mv$.

(c) Solve the two equations from a) and b) to find the possible values of the orbiting radius, r_n.

(d) What are the possible energy values, E_n, of the He^+ ion?

17. **[Fall 2010, Final]** A particle of mass m is trapped in one-dimensional box of length L, as shown in Figure 29.2. The potential energy is 0 inside the box (that is, $V(x) = 0$ at $0 < x < L$), and is very large (infinite) outside the box, so that the particle cannot be outside the box at all.

    (a) Thinking of the wave function of the particle as a sine wave, with fixed ends, find the wavelengths λ_n that are allowed in this problem.

    (b) Use the de Broglie relation $\lambda_n = h/p_n$ to find the allowed momenta.

    (c) Find the allowed energies E_n.

    (d) What is the frequency of a photon that is emitted as the particle drops from level $n = 5$ to level $n = 2$?

    (e) Find the lowest total energy of three electrons placed in this potential.

18. **[Spring 2011, Final]** Bohr Model for a proton in the nucleus: A proton is moving inside a nucleus. The potential energy is $U(r) = \frac{1}{2}kr^2$. Here r is the distance from the center of the nucleus.

    (a) What is the magnitude and direction of the force on the proton?

    (b) Write the equation of motion $F = ma$ assuming the proton is in a circular orbit of radius r.

    (c) Write Bohr's postulate as a relation between the proton wavelength λ and the circumference of the orbit. Substitute the de Broglie relation $\lambda = \dfrac{h}{mv}$ to obtain Bohr's postulate in terms of the velocity v and the radius r.

    (d) Solve the two equations from b) and c) to find the values of r.

    (e) What are the possible energy values?

19. **[Spring 2011, Final]** An electron of mass m is oscillating in a molecule. The potential energy is given as $U(x) = \frac{1}{2}m\omega^2x^2$, where ω is the oscillation frequency.

    (a) Use the Uncertainty Relation to write an estimate of the total energy in terms of x.

    (b) For what value of Ex is the energy minimum?

    (c) What is the minimum energy?

Answers to Selected Problems

Chapter 4

2. (a) $[G] = L^3M^{-1}T^{-2}$;
   $[\hbar] = L^2M^1T^{-1}$;
   $[c] = L^1M^0T^{-1}$
   (b) $x = -1/2$,
   $y = 1/2, \quad z = 1/2$
   (c) $m_{PL} = 2.18 \times 10^{-8}$ kg $\gg m_p$
   (d) $t_{PL} = \sqrt{\hbar G/c^5} = 5.39 \times 10^{-44}\,\mathrm{s}$
   (e) $l_{PL} = \sqrt{\hbar G/c^3} = 1.62{\times}10^{-35}\,\mathrm{m}$

3. $[G] = L^3M^{-1}T^{-2}$;
   $m^3kg^{-1}s^{-2}$

4. $[k] = L^0M^1T^{-2}$; $kg\ s^{-2}$

6. $[K] = L^3M^1T^{-2}Q^{-2}$;
   $m^3kg^1s^{-2}C^{-2}$

   $[\epsilon_0] = L^{-3}M^{-1}T^2Q^2$;
   $m^{-3}kg^{-1}s^2C^2$

7. Flow Rate $= 4.0 \times 10^3$ tons/s
   Flux $= 8.0 \times 10^4$ kg/s/m^2

8. (a) 0
   (b) $|\mathbf{P_1}| = 3$, $|\mathbf{P_2}| = 3$
   (c) 90°
   (d) (6, 6, 3)
   (e) $(-6, -6, -3)$

10. (a) $(4.8 \times 10^{-6})(-\mathbf{j} + \mathbf{k})\,\mathrm{N}$
    (b) $6.8 \times 10^{-6}\,\mathrm{N}$

12. (a) $\mathbf{v}_{\rm p,g} = 120\mathbf{i}$ km/h; $|\mathbf{v}_{\rm p,g}| = 120$ km/h
    (b) $\mathbf{v}_{\rm p,g} = (100\mathbf{i} - 20\mathbf{j})$ km/h; $|\mathbf{v}_{\rm p,g}| = 102$ km/h
    (c) $\mathbf{v}_{\rm p,g} = (110\mathbf{i} + 5\mathbf{j})$ km/h; $|\mathbf{v}_{\rm p,g}| = 110$ km/h

Chapter 5

3. (a) $v(t) = (20 - 10t)\,\mathrm{m/s}$
   (b) $a(t) = -10\,\mathrm{m/s^2}$
   (c) $h(1s) = 15\,\mathrm{m}$; $\quad h(2s) = 20\,\mathrm{m}$; $h(3s) = 15\,\mathrm{m}$
   $v(1s) = 10\,\mathrm{m/s}$; $\quad v(2s) = 0\,\mathrm{m/s}$; $\quad v(3s) = -10\,\mathrm{m/s}$
   $a(1s) = -10\,\mathrm{m/s^2}$; $\quad a(2s) = -10\,\mathrm{m/s^2}$; $\quad a(3s) = -10\,\mathrm{m/s^2}$

7. (a) $(5\cos\theta\mathbf{i} + (5\sin\theta - 10t)\mathbf{j})$ m/s
   (b) 5 m/s
   (c) $-10\mathbf{j}$ m/s^2
   (d) $x\tan\theta - x^2/5\cos^2\theta$
   (e) $\sqrt{25 - 100t\sin\theta + 100t^2}$ m/s

Chapter 6

2. (a) $v(t) = (-10t + 10)\,\mathrm{m/s}$
   (b) $z(t) = (-5t^2 + 10t + 3)\,\mathrm{m}$
   (c) $z(1s) = 8\,\mathrm{m}$; $\quad v(1s) = 0\,\mathrm{m/s}$
   $z(2s) = 3\,\mathrm{m}$; $\quad v(2s) = -10\,\mathrm{m/s}$

5. (a) $t_{\max} = v_0\sin\theta/g$
   (b) $h_{\max} = v_0^2\sin^2\theta/2g$
   (c) $D = v_0^2\sin 2\theta/g$
   (d) $\theta = Dg/2v_0^2$
   (e) $h = D^2g/8v_0^2$
   (f) 2 cm

9. (a) $a = 10^3\,\mathrm{m/min^2}$

(c) 1.1×10^4 m

(d) $v(t) = 10^3 t$ m/min (if $0 \le t \le$ 1 s)

$v(t) = 10^3$ m/min (if $1 < t \le$ 11 s)

$v(t) = -10^3 t + 1.2 \times 10^4$ m/min (if $11 < t \le 12$ s)

(e) 1.1×10^4 m

10. (a) $\mathbf{v}(5s) = 112.5\mathbf{i} + 5\mathbf{j}$

(b) $v(5s) = 113$ m/s

(c) $a(5s) = 7.8$ m/s^2

Chapter 7

3. (a) $r\cos\omega t\mathbf{i} + r\sin\omega t\mathbf{j} + z_0(1 + (1/2)\sin 3\omega t)\mathbf{k}$

(b) $-r\omega\sin\omega t\mathbf{i} + r\omega\cos\omega t\mathbf{j} + z_0(3\omega/2)\cos 3\omega t\mathbf{k}$

(c) $-r\omega^2\cos\omega t\mathbf{i} - r\omega^2\sin\omega t\mathbf{j} - z_0(9\omega^2/2)\sin 3\omega t\mathbf{k}$

(d) $\sqrt{r^2\omega^2 + (9\omega^2/4)z_0^2\cos^2 3\omega t}$

4. (a) $P = 3.16 \times 10^7$ s (= 1 year)

(b) $v = 3.0 \times 10^4$ m/s

(c) $a = 6.0 \times 10^{-3}$ m/s^2, towards the Sun

(d) $r(t) = R(\cos(\omega t)\mathbf{i} + \sin(\omega t)\mathbf{j})$, where $R = 1.5 \times 10^{11}$ m
$v(t) = \omega R(\cos(\omega t)\mathbf{i} + \sin(\omega t)\mathbf{j})$
$a(t) = -\omega^2 R(\cos(\omega t)\mathbf{i} + \sin(\omega t)\mathbf{j})$

5. (a) 200 m

(b) 4.9 m/s^2

(c) Towards the center of the circular path

(d) 4.4 m/s^2, towards the center of the circular path

Chapter 8

1. $v_{\min} = \sqrt{gR}$

2. (a) $F_N = mv^2/R - mg\cos\alpha$

(b) $v = \sqrt{gR\cos\alpha}$

4. (b) 0

(c) $mg\cos\theta$

5. (a) $t_{inc}/t = \sqrt{2}$

(b) $t_{inc}/t \sim 1/\alpha$

6. (a) equal to W_g

(b) equal to W_g

(c) less than W_g

(d) greater than W_g

(e) greater than W_g

(f) less than W_g

(h) $F_{\text{net}} = ma$

(i) Yes.

(j) You feel weightless ($F_N = 0$)

9. $\Sigma F = Mg\sin\theta - \mu Mg\cos\theta$
$a = g(\sin\theta - \mu\cos\theta)$

Chapter 9

1. $\mathbf{v}_{1,\mathrm{f}} = (10\mathbf{i} + 10\mathbf{j})$ m/s, at $+45^{\circ}$

$\mathbf{v}_{2,\mathrm{f}} = (10\mathbf{i} - 10\mathbf{j})$ m/s, at -45°

2. (a) $\mathbf{v}_{\mathrm{f}} = 6.6 \times 10^2$ m/s, in $+x$ direction

(b) $\mathbf{F}_{\mathrm{j,r}} = 6 \times 10^4$ N in $+x$ direction

5. (a) $\mathbf{r}_1(\mathrm{t}) = (3\mathrm{t} + 1)\,\mathbf{i} + 3\mathrm{t}\,\mathbf{j}$ m

$\mathbf{r}_2(\mathrm{t}) = (3\mathrm{t} + 1)\,\mathbf{i} + (-3\mathrm{t} + 6)\,\mathbf{j}$ m

(b) $\mathbf{r}_{\mathrm{cm}}(\mathrm{t}) = (3\mathrm{t} + 1)\,\mathbf{i} + (\mathrm{t} + 2)\,\mathbf{j}$ m

(c) $\mathbf{V}_{\mathrm{cm}} = (3\,\mathbf{i} + \mathbf{j})$ m/s

(e) $\mathbf{v}_{1,\mathrm{cm}} = (2\,\mathbf{j})$ m/s

$\mathbf{v}_{2,\mathrm{cm}} = (-4\,\mathbf{j}\,)$ m/s

(f) $\mathbf{p}_{1,\mathrm{cm}} = (4\,\mathbf{j}\,)$ kg m/s

$\mathbf{p}_{2,\mathrm{cm}} = (-4\,\mathbf{j}\,)$ kg m/s

(g) $\mathbf{r}_{1,\mathrm{cm}} = (2\mathrm{t} - 2)\,\mathbf{j}$ m

$\mathbf{r}_{2,\mathrm{cm}} = (-4\mathrm{t} + 4)\,\mathbf{j}$ m

(j) at t = 1 s.

(k) 27 J

Chapter 10

1. They all have same speed.
2. $v_3 > v_2 > v_1$
3. Work $= -\mu\, mgh/\tan\theta$;
   KE $= mgh(1 - \mu/\tan\theta)$

Chapter 12

1. (a) 0.1 m
   (b) Yes.
   (d) $F(x = 0.2m) = -160$ N, towards equilibrium point
   $F(x = -0.2m) = -80$ N, away from equilibrium point
5. (a) $\omega t = \omega t_0 + 2\pi$
   (b) P $= 2\pi/\omega$
   (c) 20 cycles per second
   $\omega = 40\pi$ /s
   (d) f $= \omega/2\pi$
8. $A = R\cos\phi$; $B = -R\sin\phi$
   $A = R'\sin\phi'$; $B = R'\cos\phi'$
10. $$v(t) = -\omega A\sin(\omega t) + \omega B\cos(\omega t)$$
    $$= -\omega R\sin(\omega t + \phi)$$
    $$= \omega R'\cos(\omega t + \phi')$$
11. (a) $\omega = 2/\mathrm{s}$
    (b) $P = \pi\,\mathrm{s}$
    (c) KE $= 8\cos^2(\omega t + \pi/6)\,\mathrm{J}$
    (d) PE $= 8\sin^2(\omega t + \pi/6)\,\mathrm{J}$
    (e) ΣE $= 8\,\mathrm{J}$
12. (a) $\omega = 10\,\mathrm{s}^{-1}$; $P = 0.63\,\mathrm{s}$
    (b) $t_0 = 0, \pi/\omega\,\mathrm{s}$
    (c) $t_{\max_+} = 3\pi/2\omega\,\mathrm{s}$
    (d) $t_{\max_-} = \pi/2\omega\,\mathrm{s}$
    (e) $\Omega(t) = -\theta_0\omega\sin(\omega t + \pi/2)$
    (f) $\Omega(t_0) = -\theta_0\omega$ or $\theta_0\omega$; $\Omega(t_{\max_+}) = 0$; $\Omega(t_{\max_-}) = 0$

Chapter 13

1. (a) $L = 2\pi m r^2/P$

Chapter 14

2. (a) $a = 2.56 \times 10^{-3}\,\mathrm{m/s^2}$
   (b) $a_{\mathrm{moon}}/a_{\mathrm{apple}} = 2.61 \times 10^{-4} \sim 1/3600$
   (c) $a_{\mathrm{moon}}/a_{\mathrm{apple}} = 2.84 \times 10^{-4}$

Chapter 15

1. (a) $n = 6 \times 10^{23}\,/\mathrm{m}^3$
   (b) $d \sim 10^{-8}\,\mathrm{m} \gg 10^{-10}\,\mathrm{m}$; Dilute gas
3. (a) Yes. They are in thermal equilibrium.
   (b) $N = 2.4 \times 10^{25}$ molecules
   (c) $n_{\mathrm{O_2}} = 4.8 \times 10^{24}\,/\mathrm{m}^3$
   (d) $n_{\mathrm{N_2}} = 1.9 \times 10^{25}\,/\mathrm{m}^3$
   (e) $P_{\mathrm{O_2}} = 2.0 \times 10^4\,\mathrm{Pa}$;
   $P_{\mathrm{N_2}} = 8.0 \times 10^4\,\mathrm{Pa}$
4. (a) $v_{\mathrm{rms,\,O_2}} = 484\,\mathrm{m/s}$;
   $v_{\mathrm{rms,\,N_2}} = 517\,\mathrm{m/s}$
   (b) $v_{\mathrm{rms}} \propto 1/\sqrt{m}$
5. (a) $n = 7.4 \times 10^{-2}\,\mathrm{mol}$
   (b) $T_f = 325\,\mathrm{K}$

Exam Problems – Thermodynamics

1. (a) $T_i = 300\,\mathrm{K}$; $T_i = 1200\,\mathrm{K}$
   (b) $\Delta N = 3/4\ N_i$
   (c) $v_{rms,\,f}/v_{rms,\,i} = 2$
   (d) The total internal energy remains the same

2. (a) $kT = 2/3\ \langle KE \rangle$
   (b) The internal energy remains constant
   (c) $W = 8300(\ln 3)$ J
   (d) The heat removed from the gas

Chapter 16

1. (a) 6.3 $\times 10^{18} e^-$
   (b) $N_{\text{atom}} \sim 10^{24}$ atoms.
   (c) for 1 cm^3, 10^{-5}

3. $\mathbf{F}_{\text{E}} = 9.2 \times 10^{-8}\,\text{N}\;\hat{e_r}$
   $\mathbf{F}_{\text{G}} = 4.1 \times 10^{-47}\,\text{N}\;\hat{e_r} \quad \ll \mathbf{F}_{\text{E}}$

4. (b) $\mathbf{E}(0{,}0) = 0$

5. $\mathbf{E}_{\text{A}} = 4k\,\mathbf{k}$ N/C (where k is Coulomb's constant)
   $\text{E}_{\text{B}} = \text{E}_{\text{D}} = \text{E}_{\text{A}}$
   $\mathbf{E}_{\text{C}} = 0$

7. (a) points up
   (b) $\mathbf{E}_0 = 0$ (they DO cancel out when you calculate)

8. (a) 0
   (b) $\perp$ to surface, outwards
   (c) $\perp$ to surface, inwards

12. (a) vertically upward
    (b) vertically downward
    (c) $\text{E} = -m_e g/e$
    (d) 2.8 $\times 10^{-11}$ V;
    $\text{V}_+ > \text{V}_-$

Chapter 17

1. (a) $\Phi_{\text{total}} = 0$

2. (a) $\text{E}_{r<R} = \rho\, r/3\epsilon_0$;
   $\text{E}(r = 5\,\text{cm}) = 1.9 \times 10^{11}\,\text{NC}^{-1}$
   (b) $\text{E}_{r>R} = \rho\, R^3/3\epsilon_0\, r^2$
   $\text{E}(r = 10\,\text{cm}) = 3.8 \times 10^{11}\,\text{NC}^{-1}$;
   $\text{E}(r = 20\,\text{cm}) = 9.4 \times 10^{10}\,\text{NC}^{-1}$
   (d) $\text{V}_{r<R} = -\rho\, r^2/6\epsilon_0$
   $\text{V}_{r<R} = \rho\, R^2/3\epsilon_0(\frac{R}{r} - \frac{3}{2})$

3. (a) $-0.4\,\text{C/m}^2$
   (b) $\text{E}_{r<R} = 5/4\pi\epsilon_0 r^2$; radially outwards
   $\text{E}_{r>R} = 0$

Chapter 18

3. (a) 3×10^{-13} F
   (b) $10^2\,\text{C}\,\text{m}^{-2}$
   (c) $10^{13}\,\text{N}\,\text{C}^{-1}$
   (d) 3×10^{10} V

4. (a) $\mathbf{E}_{r<R} = kQ/r^2\,\hat{r}$; where $k = 1/4\pi\epsilon_0$
   (b) $\Delta V = kQ/R$
   (c) $C = R/k = 4\pi\epsilon_0 R$

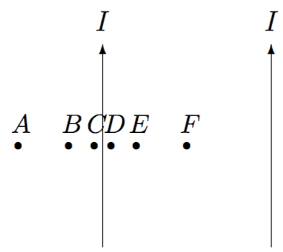
Exam Problems – Gauss' Law

4. (a) $\mathbf{E}_{\,r>b} = 0$
   (b) $\mathbf{E}_{\,a<r<b} = Q/4\pi\epsilon_0 r^2\,\hat{r}$
   (c) $\mathbf{E}_{\,r<a} = Q\,r/4\pi\epsilon_0 a^3\,\hat{r}$
   (d) $\Delta V_{ab} = Q/4\pi\epsilon_0(\frac{1}{a} - \frac{1}{b})$

Chapter 19

2. (a) $N_{\text{carriers}} = nAvt$;
   $Q = qnAvt$
   (b) $I = qnAv$
   (c) $j = qnV$

4. Take the current I upwards, and let out of paper + & into the paper − A & F is 5 cm away from the left current. B & E is 2 cm away, and C &



$B_A = 4.2 \times 10^{-6}$ T;
$B_D = -1.8 \times 10^{-5}$ T
$B_B = 1.2 \times 10^{-5}$ T;
$B_E = -7.5 \times 10^{-6}$ T
$B_C = 2.2 \times 10^{-5}$ T;
$B_F = 0$

7. To the right

8. Top two: Force is to the right
   Bottom two: Force is to the left

9. (a) Downward
   (b) Because the force is always perpendicular to **v**
   (c) $r = mv/qB$

12. (a) $I = nev_0 A$
    (b) $\mathbf{F} = nev_0 ALB_0\,\mathbf{k}$

Chapter 21

1. (a) $B_{r<R} = \mu_0 Ir/2\pi R^2$
   (b) $B_{r>R} = \mu_0 I/2\pi r$

Chapter 22

1. (a) $-\mu_0 nA\frac{dI}{dt}$
   (b) $-\mu_0\frac{N^2}{l}A\frac{dI}{dt}$
   (c) $\mu_0\frac{N^2}{l}A$
   (d) $\mathrm{kg\,m^2\,C^{-2}}$

2. $L = 1.3\times 10^{-5}\,\mathrm{H}$
   $V(t) = -7.9\times 10^{-3}\,\cos(100\pi t)\,\mathrm{V}$

3. $V(t) = -3.1\times 10^{-7}\cos(100\pi t)\,\mathrm{V}$

Chapter 23

2. (a) $E_0 = V_0/D = 15\,\mathrm{V/m}$
   (b) $B_0 = -E_0/c = -5\times 10^{-8}\,\mathrm{T}$

3. (a) **E** in the direction of the current, and **B** in circles around the current according to the right hand rule.
   (b) The EM wave propagates radially out from the current-carrying wire.
   (c) $\omega = 100\pi\,/\mathrm{s}$
   (d) $k = 10^{-6}\,/\mathrm{m}$
   $\lambda = 6\times 10^6\,\mathrm{m}$

4. (a) For $f = 100\,\mathrm{MHz}$,
   $\lambda = 3\,\mathrm{m}$
   (b) For $f = 10\,\mathrm{MHz}$,
   $\lambda = 30\,\mathrm{m}$
   (c) For $f = 60\,\mathrm{MHz}$,
   $\lambda = 5\,\mathrm{m}$

Exam Problems – E&M

3. (a) $\Phi_B = B(\pi a^2)\sin(\omega t)$, if the ring is on x-y plane at $t = 0$ ($\mathbf{dS} = \mathrm{dS}\,\mathbf{k}$ initially)
   (b) $V = -\omega B(\pi a^2)\cos(\omega t)$

4. Take the coordinates as follows: +x into the page, +y to the left, +z upwards
   (a) **E** perpendicular to the plates, with the directions alternating depending on $V(t)$ ($\pm z$).
   (b) **B** along the x-axis with alternating directions, depending on **E** directions ($\pm x$).
   (c) EM waves propagates in +y direction, towards the infinite side.
   (d) $\lambda = 10^2\,\mathrm{m}$

8. (a) $E = V/D$ in the direction of + to −
   (b) $\mathbf{F} = -e\mathbf{E}$ (from − to +)
   (c) $B = V/v_e D$, into the page

Chapter 25

5. $v_1 = 2.2\times 10^6\,\mathrm{ms^{-1}}$
   $v_2 = 1.1\times 10^6\,\mathrm{ms^{-1}}$
   $v_3 = 7.3\times 10^5\,\mathrm{ms^{-1}}$

6. (a) $E_1 = -E_0\alpha^2/2$
   (b) $a_0 = \lambda_e/\alpha$

7. (a) $f = 3.3\times 10^{15}\left[\frac{2n-1}{n^2(n-1)^2}\right]\mathrm{s^{-1}}$
   $\omega = 2.1\times 10^{16}\left[\frac{2n-1}{n^2(n-1)^2}\right]\mathrm{s^{-1}}$
   (b) $ke^2/\hbar r_n n$

Chapter 26

1. $v = h/\lambda m$

2. (a) $p = 9.1 \times 10^{-31}\,\mathrm{kg\,ms^{-1}}$; $\lambda = 7.3 \times 10^{-4}\,\mathrm{m}$
   (b) $p = 9.1 \times 10^{-28}\,\mathrm{kg\,ms^{-1}}$; $\lambda = 7.3 \times 10^{-7}\,\mathrm{m}$
   (c) $p = 9.1 \times 10^{-25}\,\mathrm{kg\,ms^{-1}}$; $\lambda = 7.3 \times 10^{-10}\,\mathrm{m}$

3. (a) $2\times 10^{10}\,\mathrm{m^{-1}} \leq k \leq 4\times 10^{10}\,\mathrm{m^{-1}}$
   (b) $1.7 \times 10^{6}\,\mathrm{ms^{-1}}$
   (c) $3.4 \times 10^{6}\,\mathrm{ms^{-1}}$
   (d) $3.2 \times 10^{-24}\,\mathrm{kg\,ms^{-1}}$

5. (a) for $m = 50$ kg; $6.6 \times 10^{-36}\,\mathrm{m}$
   (b) $1.3 \times 10^{-36}\,\mathrm{m}$

6. (a) $p^2/2m_e$
   (b) $\hbar^2 k^2/2m_e$
   (c) hf

7. (a) $E = \sqrt{p^2c^2 + m^2c^4}$
   (b) $E = \sqrt{\hbar^2k^2c^2 + m^2c^4}$
   (c) $E = hf$

Chapter 27

3. (a) $z = (\hbar^2/m^2 g)^{1/3}$
   $E_{\min} = \frac{3}{2}\,(\hbar^2 m g^2)^{1/3}$
   (b) for electron: $z = 1.1 \times 10^{-3}\,\mathrm{m}$
   for proton: $z = 7.3 \times 10^{-6}\,\mathrm{m}$
   for $m = 100\,\mathrm{kg}$: $z = 4.8 \times 10^{-25}\,\mathrm{m}$

Chapter 28

1. (a) $f_-(x_1, x_2) = \dfrac{2}{L}\left[\sin\left(\tfrac{3\pi}{L}x_1\right)\sin\left(\tfrac{4\pi}{L}x_2\right) - \sin\left(\tfrac{4\pi}{L}x_1\right)\sin\left(\tfrac{3\pi}{L}x_2\right)\right]$
   (b) $f_+(x_1, x_2) = \dfrac{2}{L}\left[\sin\left(\tfrac{1\pi}{L}x_1\right)\sin\left(\tfrac{5\pi}{L}x_2\right) + \sin\left(\tfrac{5\pi}{L}x_1\right)\sin\left(\tfrac{1\pi}{L}x_2\right)\right]$

2. (a) $E = (42.5)\dfrac{\hbar^2\pi^2}{mL^2}$
   (b) $E = (4.5)\dfrac{\hbar^2\pi^2}{mL^2}$

Chapter 29

1. $1s^2\ 2s^2\ 2p^5$

Exam Problems – Quantum Mechanics

2. (a) $E = \hbar^2/(8mx^2) + Cx^4$
   (b) $x_{\min} = (\hbar^2/16\,Cm)^{1/6}$
   (c) $E_{\min} = \dfrac{3}{4}\left(\dfrac{\hbar^4 C}{4m^2}\right)^{1/3}$
   (d) $KE = \dfrac{1}{2}\left(\dfrac{\hbar^4 C}{4m^2}\right)^{1/3}$
   (e) $U = \dfrac{1}{4}\left(\dfrac{\hbar^4 C}{4m^2}\right)^{1/3}$

4. (a) $\lambda = 2L/n$
   (b) $p_n = nh/2L$
   (c) $E_n = n^2h^2/8L^2m$
   (d) $f = 3h/2mL^2$

5. (a) $\dfrac{1}{4\pi\epsilon_0}\dfrac{3e^2}{r^2} = \dfrac{mv^2}{r}$
   (b) $2\pi r = n\lambda = nh/mv$
   (c) $r = 4\pi\epsilon_0 n^2\hbar^2/3me^2$
   (d) $E = -\dfrac{1}{2}\dfrac{9me^4}{(4\pi\epsilon_0)^2 n^2\hbar^2}$

8. (a) $F = 3Cr^2$
   (b) $3Cr^2 = mv^2/r$

$2\pi r = nh/mv$

(c) $r = (n^2\hbar^2/3Cm)^{1/5}$

(d) $E = \frac{5}{6}\left(\frac{n^6\hbar^6 9C^2}{m^3}\right)^{1/5}$

9. (a) $r = n^2\hbar^2/ke^2m$

(b) $E = -\frac{1}{2}\frac{k^2e^4m}{n^2\hbar^2}$

(c) $E = 5E_1/36$;
$f = 5E_1/36h$

14. (a) $v_n = \hbar/nma_0$

(b) $KE = \hbar^2/2mn^2a_0^2$

(c) $E = \frac{1}{n^2}\left(\frac{\hbar^2}{2ma_0^2} - \frac{e^2}{4\pi\epsilon_0 a_0}\right)$

(d) $f = \frac{8}{9h}\left(\frac{e^2}{4\pi\epsilon_0 a_0} - \frac{\hbar^2}{2ma_0^2}\right)$

Glossary

acceleration = *ivme*
angular speed / velocity = *açısal hız*
arc = *yay*
beat = *vuru*
blackbody radiation = *karacisim ışıması*
boundary conditions = *sınır koşulları*
to brake = *frenlemek*
bound = *bağlı*
capacitance = *sığa*
capacitor = *sığa, kapasitör, kondansatör*
circuit = *devre*
circulation = *dolanma*
circumference = *çember*
collective motion = *toplu hareket*
collision = *çarpışma*
comet = *kuyrukluyıldız*
compass = *pusula*
component = *bileşen*
composite = *bileşik*
conducting = *iletken*
conservation laws = *korunum yasaları*
constant = *sabit*
contact = *temas*
continuum (plural: continua) = *sürekli ortam*
coupled = *etkileşen*
to damp = *frenlemek, etkisini azaltmak*
degree = *derece*
derivative = *türev*
derivative of x with respect to t = x*'in* t*'ye göre türevi*
diameter = *çap*
dice (plural) = *oyun zarlaı*
die (singular)= *zar*
diffraction = *kırınım*
Earth = *Dünya, Yerküre*
exponential function = *üssel fonksiyon*
equation of motion = *hareket denklemi*

equilibrium point = *denge noktası*
flux = *akı*
ground (voltage) =*toprak (voltaj)*
heat = *ısı*
hypothenus = *hipotenüs*
impedance = *empedans, direnç, çeli*
inductor = *indüktör*
inertia = *eylemsizlik, atalet*
interference = *girişim*
insulating = *yalıtkan*
latitude = *enlem*
length of arc = *yay uzunluğu*
longitude = *boylam*
momentum = *momentum*
nucleus (plural: nuclei) = *çekirdek, atom çekirdegi*
period = *peryod, dönem*
perpendicular = *dik*
phase = *faz*
pole = *kutup*
projectile motion = *eğik atış*
projection = *izdüşüm*
propagation = *yayılma, dalganın ilerlemesi*
radian = *radyan*
radian velocity = *açısal hız*
radius = *yarıçap*
resistance = *direnç*
resistivity = *özdirenç*
resistor = *direnç*
response = *tepki*
revolution = *tam dönüş*
right angle = *dik açı*
rigid = *katı*
scale = *ölçek*
sink = *delik, lavabo deliği*
solenoid = *bobin*
source = *kaynak*
species = *canlı türleri*
speed = *sürat*
stable equilibrium points = *kararlı denge noktaları*
tangent = *teğet*
threshold = *eşik*
transparent = *şeffaf*
uniform = *tekdüze*
velocity = *hız*
Voltaic battery = *Volta pili, pil*

wave equation= *dalga denklemi*
wave packet = *dalga paketi*
weight = *ağırlık*
work = *iş*